


Lecture 1. $\mathcal{L}u = \sum \underbrace{a_{ij}}_{\text{对称的}} u_{ij} + \sum b_i u_i + cu = f$ in $\Omega \subseteq \mathbb{R}^n$

Elliptic. $(a_{ij})(x) > 0$ 正定 $\forall x \in \Omega$

uniformly elliptic $\underline{\lambda}I \leq (a_{ij}(x)) \leq \underline{\Lambda}I$ $0 < \lambda \leq \Lambda \quad \forall x \in \Omega$

Examples. $\Delta u = 0$

"足够简单, 足够复杂." 运用更多 case.

Consider $a_{ij} u_{ij} = f \quad 0 < \lambda I \leq (a_{ij}(x)) \leq \Lambda I \quad \forall x \in \Omega$

✓ • Schauder estimates (1930s) $C^\alpha = \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\alpha} < \infty$
 $a_{ij} \in C^\alpha(\bar{\Omega}), f \in C^\alpha(\bar{\Omega}) \Rightarrow u \in C^{2,\alpha}(\bar{\Omega})$

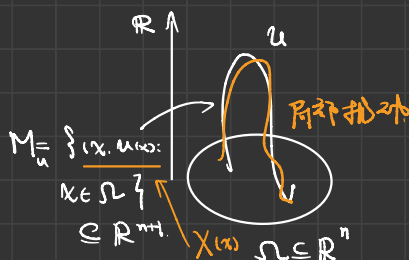
系数足够好, 方程正则性就好!

• Calderon-Zygmund estimates (1950s)

$a_{ij} \in C^0(\bar{\Omega}), f \in L^p(\bar{\Omega}) \quad (p > 1) \rightsquigarrow u \in W^{2,p}(\bar{\Omega})$

Assume orthogonal.

$$\begin{aligned} \text{Area}(M_u) &= \int_{\Omega} \sqrt{\det g_{ij}} \\ &= \int_{\Omega} \sqrt{1 + |\nabla u|^2} \end{aligned}$$



Define: $F(\xi) = \sqrt{1 + |\xi|^2}$

$$\text{Area}(M_u) = \int_{\Omega} F(\nabla u)$$

$$g_{ij} = \delta_{ij} + u_i u_j$$

$$\begin{aligned} g_{ij} &= \langle X_i, X_j \rangle \\ &= \langle e_i + u_i e_{n+1}, e_j + u_j e_{n+1} \rangle \end{aligned}$$

极小曲面

$$0 = \frac{d}{dt} \Big|_{t=0} \text{Area}(M_{u+t\phi}) = \frac{d}{dt} \Big|_{t=0} \int_{\Omega} F(\nabla u + t \nabla \phi)$$

$$\forall \phi \in C_c^\infty(\Omega) \quad = \int_{\Omega} F_{g_i}(\nabla u) \phi_i = - \int_{\Omega} [\partial_i F_{g_i}(\nabla u)] \phi$$

$$\Rightarrow \boxed{\partial_i (F_{g_i}(\nabla u)) = 0} \quad \text{, 正則性如何?}$$

$$\boxed{F_{g_i g_j}(\nabla u) u_{ij} = 0} \Leftrightarrow F_{g_i g_j} = \frac{1}{\sqrt{1+|g|^2}} \left(\delta_{ij} - \frac{g_i g_j}{1+|g|^2} \right)$$

Bounded, uniformly elliptic.

We hope $u \in C^{1,\alpha}$ to apply Schauder estimates

$$\text{Suppose } u \in C^{1,\alpha} \xrightarrow[\text{Schauder estimate}]{F_{g_i g_j} u_{ij} = 0} u \in C^{2,\alpha}$$

$$\text{正則性, } \underbrace{F_{g_i g_j}(\nabla u)}_{C^\alpha} \underbrace{(u_e)_j}_{C^\alpha} + \underbrace{F_{g_i g_j g_k}(\nabla u)}_{C^\alpha} \underbrace{u_{ij}}_{C^\alpha} \underbrace{(u_e)_k}_{C^\alpha} = 0 \quad \text{"e 正則"}$$

$$\xrightarrow[\text{Schauder estimate}]{\text{Schauder}} u \in C^{2,\alpha} \rightsquigarrow u \in C^{3,\alpha} \xrightarrow[\text{bootstrap argument}]{\text{bootstrap argument}} u \in C^{k,\alpha} \forall k$$

$$\text{Fix } e \in S^{n-1}. \quad \partial_i (F_{g_i g_j}(\nabla u) \partial_j |u_e|) = 0$$

✓ • De Giorgi (1957?) - Moser (1960?) | Nash (parabolic 1958?)

$$\text{"divergence"} \quad \partial_i (a_{ij}(x) \partial_j u) = 0. \quad a_{ij} \in L^\infty \Rightarrow u \in C^\alpha(\Omega)$$

(?) Krylov - Safarov (1980?)

$$a_{ij}(x) u_{ij}(x) = 0, \quad a_{ij} \in L^\infty(\Omega) \Rightarrow u \in C^\alpha(\Omega)$$

$$\bullet F(\partial^2 u) = 0 \quad (\text{e.g. } F(M) = \det M.)$$

$$\text{Evans - Krylov} \quad F \text{ is an convex / concave, } u \in C^{\frac{1}{2},1} \Rightarrow u \in C^{2,\alpha}$$

$$F_{ij}(\partial^{\alpha} u)_{,j} = 0. \xrightarrow{\text{Schauder estimates}} u \in C^{3,\alpha}$$

$\partial f u \in C^{3,\alpha}$

$$\Delta u = 0 \text{ in } \Omega \subseteq \mathbb{R}^n$$

Mean Value Property

$$u(x) = \int_{\partial B_r(x)} u = \int_{B_r(x)} u$$

proof: Define $\phi(r) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y) dy$

$$\text{If } \phi'(t) = 0, \text{ for } t \in (0, r),$$

then we are done.

Exercise

(using integration by parts)

$$y = rz + x, \quad z \in \partial B_1.$$

$$\int_{B_r(x)} u(y) dy = \int_0^r \left[\int_{\partial B_t(x)} u(z) dz \right] dt = u(x) \int_0^r |\partial B_t| dt = u(x) |B_r(x)| \quad \square$$

$\xrightarrow{u(x) |\partial B_t|}$

Rmk: (i) 上性良为 $\Delta u = 0$ 等价刻画. $u \in C^2$

$$(ii) \Delta u \geq 0, \quad u(x) \leq \int_{\partial B_r(x)} u \left(\leq \int_{B_r(x)} u \right)$$

$$\Delta u \leq 0, \quad u(x) \geq \dots \quad (\geq \dots)$$

sub-harmonic $\leftarrow$

super-harmonic $\leftarrow$

[Ex]:

$$\int_M u dVol \xrightarrow{\text{Riemann 体积元}} dVol = \sqrt{g} g$$

Remark: ① Let (M, g) be a complete Riemann manifold. $R_{\mu} \geq 0$

Then $u \geq 0$ + $\Delta u \leq 0 \Rightarrow u(x) \geq \frac{1}{|B_r(x)|} \int_{B_r(x)} u \, d\text{Vol}_g$, $r \in (0, \inf_j r_j)$

$\phi' = ?$ 同中取最大值原理

Dirichlet

→ 2014 年 2 月

截曲率

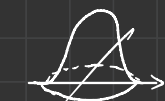
②. ----- $\text{Sec} \leq K_0$

$u \geq 0$ + $\Delta u \geq 0 \Rightarrow u(x) \leq \frac{1}{|B_r^{K_0}(x)|} \int_{B_r(x)} u \, d\text{Vol}_g$

Interior estimates

Consider $p \geq 0, \in C_c^\infty(\mathbb{R}^n)$ $\text{Supp } p = B_1$ $\int_{\mathbb{R}^n} p = 1$

$p(x) = \eta(|x|)$ Examples. $\eta(t) = \begin{cases} C e^{-\frac{1}{1-t^2}} & t \in [0, 1) \\ 0 & t \geq 1 \end{cases}$



$$\int_{\mathbb{R}^n} p(x-y) u(y) dy = \int_0^\infty \int_{\partial B_t(x)} \eta(t) u(z) dz dt$$



suppose u has the "MVP"

$$\begin{aligned} &= \int_0^\infty \eta(t) u(x) |\partial B_t(x)| dt \\ &= u(x) \int_0^\infty \eta(t) |\partial B_t(x)| dt \\ &= u(x) \int_{\mathbb{R}^n} p = u(x). \end{aligned}$$

Thm If u satisfies the mean value property in Ω

then $\forall B_r(x) \subset \Omega$ we have $|D^k u(x)| \leq \frac{C_{n,k}}{r^{n+k}} \|u\|_{L^1(B_r(x))}$

Proof: Take such p . Then $u(x) = \int_{\mathbb{R}^n} p(x-y) u(y) dy$.

$$\Rightarrow |D^k u(x)| = \left| \int_{\mathbb{R}^n} D_x^k p u(y) dy \right| \leq \frac{C_{n,k,p}}{r^{n+k}} \|u\|_{L^1(B_r(x))}$$

Consider $\tilde{u}(y) = u(\underline{ry+x})$ $y \in B_1(0)$ Check $\tilde{u}(y)$ satisfies MVP

$$|D_y^k \tilde{u}(x)| \leq C_{n,k} \|\tilde{u}\|_{L^1(B_{1/2})} = \frac{C_{n,k}}{r^n} \|\hat{u}\|_{L^1(B_r(x))}$$

$$(If \Delta u = 0) \quad \frac{1}{r^k} |D_x^k u(x)|$$

□

Corollary (Liouville) If $u \in C^\infty(\mathbb{R}^n)$, $\Delta u = 0$ and u is one-sided bounded, then $u = \text{const.}$

proof $\boxed{u \geq 0}$ By interior estimates. $\forall x_0 \in \mathbb{R}^n$

$$\begin{aligned} |Du| &\leq \frac{C_n}{r^{n+1}} \|u\|_{L^1(B_r)} \quad \forall r > 0 \\ \xrightarrow{\text{MVP}} &= \frac{C}{r} u(x) \quad \text{letting } r \rightarrow \infty \\ \Rightarrow Du(x_0) &= 0. \end{aligned}$$

$$u - \inf_{\mathbb{R}^n} u \geq 0 \quad \checkmark$$

$$\sup_{\mathbb{R}^n} u - u \geq 0 \quad \checkmark$$

□

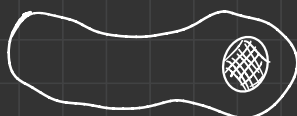
Remark: (M.g) be a complete ^{non compact} Riemann manifold, $\Delta u = 0$, $\text{Ric} \geq 0$

u is one-sided bounded $\Rightarrow u = \text{const.}$

proof: Apply maximal principle to $\eta(x) \log |Du|$

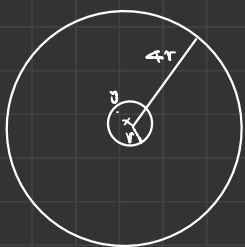
$$\frac{1}{2} \Delta |Du|^2 = |Du|^2 + \langle \nabla \Delta u, Du \rangle + \text{Ric}(\nabla u) \quad (\text{Bochner formula})$$

Thm (Maximum Principle) If $\Delta u \geq 0$, then $\max_{\Omega} u = \max_{\partial \Omega} u$

 $u(x) \leq \sup_{B_r} u$ If $\Delta u \leq 0$ then $\min_{\Omega} u = \min_{\partial \Omega} u$

Thm (Harnack inequality) $\sqrt{(u>0)}$ $\Delta u = 0$ in B_{4r} $\forall x, y \in B_r \rightarrow \Omega' \subset \subset \Omega$

$u(x) \leq C u(y)$ for some a universal $C > 0$ $\rightarrow \Omega$ (general).



$$u(y) = \int_{B_{3r}} u \geq C \int_{B_r(x)} u = C u(x)$$

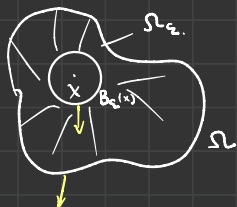
$$u(x) = \frac{1}{|B_{4r}|} \int_{B_{4r}} u \geq \frac{C}{|B_r|} \int_{B_r(x)} u = C u(x)$$

$C(n, \Omega, \Omega')$
d.l.e. Ω' ?



$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

基本解



We say $G_{\Omega}(\cdot; x_0) = \psi(\cdot; x_0) + H(\cdot)$ in Ω is Green function

$$\psi(x; x_0) = \begin{cases} -\frac{1}{2\pi} \ln |x - x_0|, & n=2 \\ \frac{1}{(n-2) |S^{n-1}| |x - x_0|^{n-2}}, & n \geq 3 \end{cases}$$

$\downarrow$ area of S^{n-1}

and H is a harmonic function

$$\begin{cases} \Delta H = 0 & \text{in } \Omega \\ H(x) = -\psi(x; x_0) & \text{on } \partial\Omega \end{cases}$$

$$\begin{cases} \Delta G_{\Omega}(x; x_0) = 0 & \forall x \in \Omega \setminus \{x_0\} \\ G_{\Omega}(x; x_0) = 0 & \text{on } \partial\Omega \end{cases}$$

$$\begin{aligned} 0 &= \int_{\Omega_{\epsilon}} u \Delta G = - \int_{\Omega_{\epsilon}} \nabla u \cdot \nabla G + \int_{\partial\Omega} u \nu G - \int_{\partial\Omega_{\epsilon}} u \nu G \\ &= \int_{\Omega_{\epsilon}} (\Delta u) G - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} G + \int_{\partial\Omega_{\epsilon}} u \nu G \\ &\quad + \int_{\partial\Omega} u \frac{\partial G}{\partial \nu} - \int_{\partial\Omega_{\epsilon}} u \frac{\partial G}{\partial \nu} \end{aligned}$$

Ω_{ϵ} is a domain with boundary $\partial\Omega_{\epsilon} = \partial\Omega \cup \partial B_{\epsilon}(x_0)$. The boundary $\partial\Omega$ is oriented outwards, and $\partial B_{\epsilon}(x_0)$ is oriented inwards. The normal vector ν on $\partial\Omega$ points outwards, and on $\partial B_{\epsilon}(x_0)$ it points inwards.

Green representation formula:

$$u(x_0) = - \int_{\Omega} f G - \int_{\partial\Omega} \varphi \partial_\nu G \quad \forall x_0 \in \Omega$$

For example B_a



定义: $\frac{1}{|x - x_0|^{n-2}} = \boxed{H}$

$$|x - x_0|^{2-n} = \left| \frac{|x_0|}{a} x - \frac{x_0}{|x_0|} a \right|^{2-n}$$

$$= \left| \frac{|x_0|}{a} \right|^{2-n} \left| x - \frac{a^2}{|x_0|^2} x_0 \right|^{2-n}$$

check $\left| \frac{|x_0|}{a} x - \frac{x_0}{|x_0|} a \right|^{2-n}$ is a harmonic func? $\checkmark$

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

Perron method

or sub-harmonic

Define: $S_\varphi = \{ \text{subsolution } v, v|_{\partial\Omega} \leq \varphi \}$

$$u(x) = \sup_{v \in S_\varphi} v(x)$$

$\hookrightarrow$ harmonic

Thm: Ω admits exterior ball condition

$$\rightarrow u|_{\partial\Omega} = \varphi$$

continuity barrier function

Q1. well-defined? $\exists \varphi \in S_\varphi$ ($S_\varphi \neq \emptyset$), $\forall v \in S_\varphi, v \leq \sup_{\partial\Omega} \varphi$

Fix $\bar{x} \in \Omega, \exists v_k \in S_\varphi, v_k(\bar{x}) \rightarrow u(\bar{x})$



$$\begin{cases} \Delta h_k = 0 & \text{in } B_s(x_k) \\ h_k = v_k & \text{on } \partial B_s(x_k) \end{cases}$$

$$\hat{v}_k = \begin{cases} h_k & \text{in } B_s(x_k) \\ v_k & \text{outside} \end{cases}$$

$H(x)$ (AA thm)

$B_{s/2}(x_k)$ [H.L.]

$$\text{WTS } \hat{v}_k(\bar{x}) \rightarrow u(\bar{x})$$

Let 2. Maximum principle

$$u \in C^2(\Omega) \cap C(\bar{\Omega})$$

Thm (weak MP) $\mathcal{L}u = a_{ij} u_{ij} + b_i u_i + c u \geq 0$ ($a_{ij} \geq \lambda \delta_{ij}, |a_{ij}|_C, |b_i|_C, |c|_C \leq \Lambda$)

Suppose $c \leq 0$ in Ω Then $\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+$

$$u^+(x) = \max \{ u(x), 0 \}$$

proof: Assume w.l.o.g $\sup_{\Omega} u \geq 0$

$$\text{Suppose } x_0 \in \Omega, u(x_0) = \sup_{\Omega} u$$

$$a_{ij} \frac{u_{ij}(x_0)}{\uparrow 0} + b_i \frac{u_i(x_0)}{\uparrow 0} + c u(x_0) = \mathcal{L}u \Big|_{x_0} > 0 \quad \text{矛盾.}$$

Consider $w = u + \varepsilon e^{\gamma x_1}$ 取 $\gamma \gg 1$ ($\delta a_{ij}, b_i, c$ 相关)

$$\mathcal{L}w = \mathcal{L}u + \varepsilon \mathcal{L}(e^{\gamma x_1}) \geq 0$$

$$\geq 0 \quad \approx (a_{11} \gamma^2 + b_1 \gamma + c) e^{\gamma x_1} \geq 0$$

$$\sup_{\Omega} u \leq \sup_{\Omega} w \leq \sup_{\partial\Omega} w$$

$$\leq \sup_{\partial\Omega} u + \varepsilon \sup_{\partial\Omega} e^{\gamma x_1} \quad \square$$

Corollary 1. $\Delta u = 0$ & $c \leq 0$, $\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u|$

proof Apply MP & u & $-u$.

Corollary 2. Suppose $c \leq 0$. if $\begin{cases} \Delta u = f \\ u = \varphi \end{cases}$ admits a $C^2(\Omega) \cap C^0(\bar{\Omega})$ solution then it is unique.

proof Apply MP $v = u_1 - u_2$.

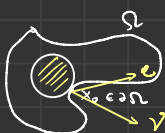
$$\begin{cases} \Delta v = 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

Thm (Hopf lemma) $\Delta u = a_{ij} u_{ij} + b_i u_i + c u \geq 0 \xrightarrow{C^2(\Omega) \cap C^0(\bar{\Omega})}$

Suppose $c \leq 0$ in Ω . If Ω admits an interior ball B_r

at $x_0 \in \partial\Omega$, and $u(x_0) > u(x) \forall x \in B_r$, $u(x_0) = \sup_{\Omega} u \geq 0$. then

$$\lim_{t \rightarrow 0^+} \frac{u(x_0) - u(x_0 - te)}{t} > 0 \quad \forall e \in S^{n-1}, e \cdot \nu > 0$$

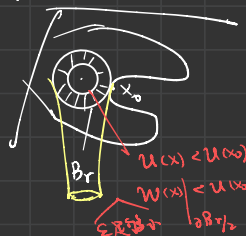


Proof. 1° construct a function h in

$$T = B_r \setminus B_{r/2} \quad \begin{cases} \Delta h \leq -1 & \text{in } T \\ h|_{\partial B_r} = 0, \quad h|_{\partial B_{r/2}} \leq 0 \end{cases}$$

$$h(x) = h(\rho), \quad \rho = |x| \in (\frac{r}{2}, r)$$

$$h' \geq 0, \quad h' \leq 0, \quad h \leq 0$$



$$\Delta h = a_{ij} h_{ij} + b_i h_i + c h$$

$$= a_{ij} \left(h' \left(\frac{\delta_{ij}}{\rho} - \frac{x_i x_j}{\rho^3} \right) + h'' \frac{x_i x_j}{\rho^2} \right) + b_i h' \frac{x_i}{\rho} + c h$$

$$\leq \lambda h'' + \frac{b_i}{\rho} \left[a_{ij} \frac{x_i x_j}{\rho^2} + b_i x_i \right] + c h$$

$$\leq \lambda h'' + 4n \lambda h' - \lambda h' = -1, \quad \text{solve} \Rightarrow h = e^{\lambda \rho}$$

$$h(\rho) = \frac{1}{\lambda} + c_1 e^{\theta_1 \rho} + c_2 e^{-\theta_2 \rho}$$

$$\theta = \theta_1, \theta_2 \text{ are roots of } \lambda \theta^2 + 4n \lambda \theta - \lambda = 0 \quad \begin{cases} h(\rho) = \frac{1}{\lambda} - c_2 e^{-\theta_2 \rho} \\ h(r) = 0. \end{cases}$$

$$\Rightarrow h(p) = \frac{1}{\Lambda} [1 - e^{-\Theta_2 (r-p)}] \quad h'(r) = \frac{1}{\Lambda} \Theta_2 > 0$$

boundary

2° Completing the proof

Consider $w = u - \varepsilon h$ in T

$$\Delta w = \Delta u - \varepsilon \Delta h \geq 0$$

Apply weak MP to w in T , $\sup_T w = \sup_{\partial T} w^+ = u(x_0)$

$$0 \leq \lim_{t \rightarrow 0^+} \frac{u(x_0) - w(x_0 - te)}{t} = \lim_{t \rightarrow 0^+} \frac{u(x_0) - u(x_0 - te) + \varepsilon h(x_0 - te)}{t}$$

$$\Rightarrow \lim_{t \rightarrow 0^+} \frac{u(x_0) - u(x_0 - te)}{t} \geq -\varepsilon \lim_{t \rightarrow 0^+} \frac{h(x_0 - te)}{t} = \varepsilon h'|_{x_0} = \varepsilon \cdot \frac{\Theta_2}{\Lambda} > 0.$$

□

Thm (Strong Maximum Principle)

Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$, $\Delta u \geq 0$

(i) If $c \leq 0$ in Ω , then $\exists x_0 \in \Omega$ s.t. $u(x_0) = \sup_{\Omega} u \geq 0 \Rightarrow u = \text{const.}$

(ii) If $c = 0$ in Ω , then $\exists x_0 \in \Omega$ s.t. $u(x_0) = \sup_{\Omega} u \Rightarrow u = \text{const.}$

Proof: Consider $\Sigma = \{x \in \Omega: u(x) = u(x_0) = \sup_{\Omega} u\}$



$y \in \Omega \setminus \Sigma$ s.t. $d(y, \partial\Omega) > d(y, \partial\Sigma)$

(i) ✓

(ii) Apply $v = u - \inf u > 0$

$$\Delta v = \Delta u \geq 0$$

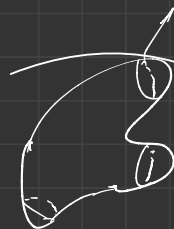
"c=0"

□

[Aleksandrov's soap theorem]

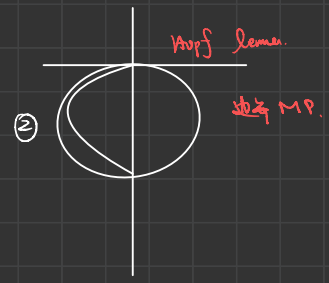
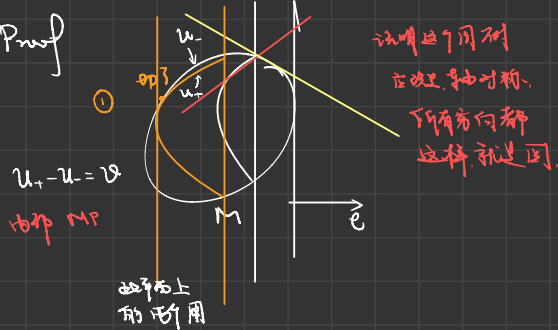
Let $M \subset \mathbb{R}^{n+1}$ be a closed embedded smooth hyper surface.

and $H_M = \text{const.}$ Then M must be a sphere.



考虑球面
处处
平均曲率
为 M 的
值 H_M
恒为 $H_M > 0$

Proof



$$u_+ \text{ satisfies } F_{ij}(Du_+)(u_+)_j = H, \quad F_{ij}(g) = \frac{1}{\sqrt{1+|g|^2}} \left(\delta_{ij} - \frac{g_i g_j}{1+|g|^2} \right)$$

$$F_{ij}(Du_-)(u_-)_j = C$$

$$\Rightarrow 0 = C - C = F_{ij}(Du_-)(u_+)_j - F_{ij}(Du_+)(u_-)_j$$

$$= \int_0^1 \frac{d}{dt} \left[F_{ij} \left(D(tu_+ + (1-t)u_-) \right) (tu_+ + (1-t)u_-)_j \right] dt$$

$$0 = \underbrace{\left(\int_0^1 F_{ij}(\dots) dt \right)}_{a_{ij}(u)} \underbrace{(u_+ - u_-)_j}_{v} + \underbrace{\left(\int_0^1 \square dt \right)}_{b_{ij}(u)} \underbrace{D_k(u_+ - u_-)}_{v_k}$$

$\therefore M, C \in \mathbb{R}$
一致有界

Apply strong mp

Thm (Aleksandrov maximum principle) (A-B-P estimates)

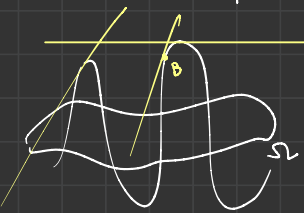
Consider $Lu := a_{ij}u_{ij} + b_i u_i + cu \geq f$ in Ω , Suppos $c \in C^0(\Omega) \cap C(\bar{\Omega})$

$a_{ij} > 0 \Rightarrow \Omega = \det a_{ij} > 0$ in Ω . $c \leq 0$ in Ω . $\frac{|b|}{\Omega^{\frac{n}{2}}}, \frac{|f|}{\Omega^{\frac{n}{2}}} \in L^n(\Omega)$

Then $\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C d \left\| \frac{f^-}{\Omega^{\frac{n}{2}}} \right\|_{L^n(\Gamma)} \rightarrow f^- = \max \{0, -f\}$

$C = C(n, \frac{|b|}{\Omega^{\frac{n}{2}}}, \frac{|f|}{\Omega^{\frac{n}{2}}})$

$$\Gamma = \Gamma_u^+ = \{x \in \Omega, u(x) \leq \underbrace{u(x) + Du(x)(y-x)}_{L_x(y)}, \forall y \in \Omega\}$$

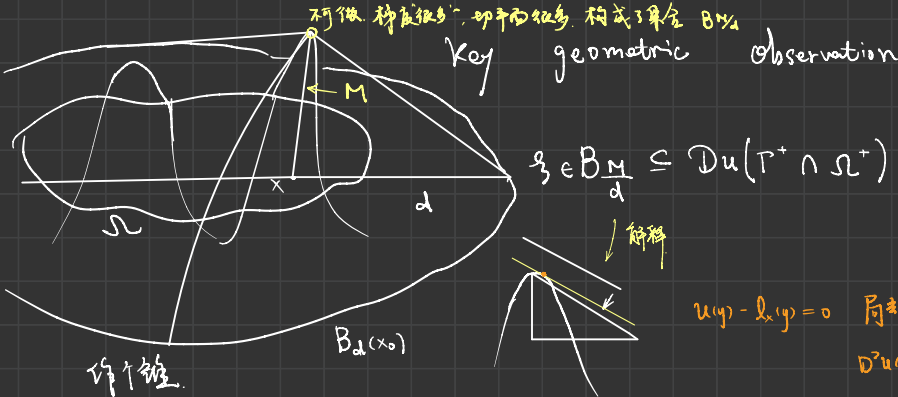


u 的图, 全部在
切平面下方 (B 就不行)

Proof: Consider $v = u - \sup_{\partial\Omega} u^+$, $v|_{\partial\Omega} \leq 0$

$$Lv = Lu - c \sup_{\partial\Omega} u^+ \geq f$$

We show $\sup_{\Omega} v \leq C d \left\| \frac{f^-}{\Omega^{\frac{n}{2}}} \right\|_{L^n(\Gamma)}$ with $u|_{\partial\Omega} \leq 0$



$$\xi \in B_{\frac{d}{2}} \subseteq Du(\Gamma^+ \cap \Omega^+) \quad \Omega^+ = \{u > 0\}$$

$$u(\eta) - L(\eta) = 0 \quad \text{局部极大值点}$$

$$D^2 u(\eta) - D^2 L(\eta) \leq 0$$

linear function

$$D^2 L = 0$$

$$A = (a_{ij})$$

$$\text{Let } g \in L^1_{loc}(\mathbb{R}^n), g \geq 0$$

$$\int_{B_{\frac{d}{2}}} g \leq \int_{Du(\Gamma^+ \cap \Omega^+)} g(\eta) d\eta \leq \int_{\Gamma^+ \cap \Omega^+} g(Du(x)) \frac{\det(-AD^2 u(x))}{\det A} \quad \left(\begin{array}{l} \because D^2 u(x) \leq 0 \\ x \in \Gamma^+ \end{array} \right)$$

$$g = Du(x) \cdot \frac{\partial g}{\partial x} = |D^2 u(x)|$$

$$\lambda_1, \lambda_2, \dots, \lambda_n \leq \left(\frac{1}{n} \sum \lambda_i \right)^n$$

$$Du(\Gamma^+ \cap \Omega^+) \text{ 与 } \Gamma^+ \cap \Omega^+ \text{ 微分同胚才行}$$

$$\text{由隐函数定理 } D^2 u(x) < 0 \text{ 才行 } u_2 = u - \frac{g}{2} |x|^2$$

$$Du_2(\Gamma^+ \cap \Omega^+) \approx Du(\Gamma^+ \cap \Omega^+)$$

$$\leq \int_{\Gamma^+ \cap \Omega^+} \frac{g(Du(x))}{2} \left| \frac{1}{n} (-a_{ij} u_{,j}) \right|^n$$

$$\begin{aligned} \Rightarrow \det(-AD^2 u(x)) &\leq C_n (-a_{ij} u_{,j})^n \\ &\leq C_n [b_i u_i + C_n \frac{f}{\mu}]^n \\ &\leq C_n (b_i u_i + f)^n \\ &\leq C_n \left((b_i f_{\mu}^-) \cdot (Du, \mu) \right)^n \\ &\leq C_n \left(|b|^n + \frac{|f|^n}{\mu^n} \right) (|Du|^n + \mu^n) \end{aligned}$$

Holder ineq

$$\int_{B_{\frac{d}{2}}} g \leq C_n \int_{\Gamma^+ \cap \Omega^+} \frac{g(Du)}{2} (|Du|^n + \mu^n) \left(|b|^n + \frac{|f|^n}{\mu^n} \right)$$

$$\text{Take } g(f) = \frac{1}{\mu^n + |f|^n} \quad (\text{局部可积的})$$

$$C_n \int_{\frac{d}{2}} \frac{t^{n-1}}{\mu^n + t^n} dt \leq C'_n \int_{\Gamma^+ \cap \Omega^+} \frac{|b|^n}{2} + \frac{(f)^n}{2 \mu^n}$$

$$\ln\left(\frac{\mu^n}{d^n \mu^n} + 1\right)$$

$$\Rightarrow M^n \leq \left[\exp \left(C \int_{\Gamma} \frac{|b|^n}{2} + \frac{|f|^n}{2 \mu^n} \right) - 1 \right] \mu^n d^n$$

$$(i) \text{ 若 } f^- = 0 \quad \text{令 } \mu \rightarrow 0$$

$$(ii) \text{ 若 } \left\| \frac{f^-}{2^n} \right\|_{L^1(\Gamma)} \neq 0, \text{ 取 } \mu = \left\| \frac{f^-}{2^n} \right\|_{L^1(\Gamma)}$$

$$\frac{1}{\mu^n} \left\| \frac{f^-}{2^n} \right\|_{L^1(\Gamma)}^n$$

≠

"Fake Proof" (等周不等式)

Isoperimetric inequality (X, Cabre)

$$\begin{cases} \Delta u = \frac{|\partial\Omega|}{|\Omega|} & \text{in } \Omega \\ \partial_\nu u = 1 & \text{on } \partial\Omega \end{cases} \quad |\partial\Omega| = \int_{\partial\Omega} \partial_\nu u =$$

$$\xi \in B_1^{(0)} \subseteq Du(\Gamma)$$

$$|B_1| = \int_{Du(\Gamma)} 1 = \int_{\Gamma} \det D^2 u \leq \int_{\Gamma} \left(\frac{1}{n} \Delta u \right)^n$$

$\searrow \left(\frac{1}{n} \frac{|\partial\Omega|}{|\Omega|} \right)^n$

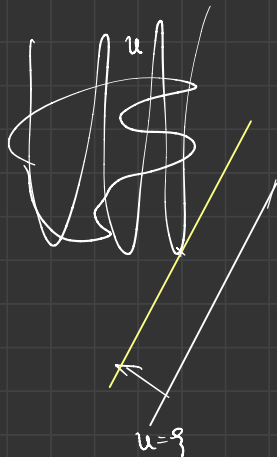
$$|B_1|^{\frac{1}{n}} \leq \left[\frac{1}{n} \frac{|\partial\Omega|}{|\Omega|} \right] |\Omega|^{\frac{1}{n}} = \frac{1}{n} \frac{|\partial\Omega|}{|\Omega|^{\frac{1}{n}}}$$

面积可以控制体积，且仅达到最佳常数(球面)

$$\Rightarrow |\partial\Omega| \geq n |B_1|^{\frac{1}{n}} |\Omega|^{\frac{1}{n}}$$

下接触集

$$\overline{T_u^-} = \{x \in \Omega : u(y) \geq u(x) + Du(x)(y-x), \forall y \in \Omega\}$$



$$\star \quad a_{ij} u_{,j} = f(x) \sqrt{1 + |Du|^2} \quad a_{ij} = \delta_{ij} - \frac{u_i u_j}{1 + |Du|^2} \quad (\text{mean curvature equation})$$

$$\int_{B_{\frac{1}{2}}} g \leq \int_{\Gamma \cap \Omega^+} g(Du) \det(-D^2 u) = \int_{\Gamma \cap \Omega^+} \frac{g(Du)}{D} \det(-AD^2 u) \leq C_n \int_{\Gamma \cap \Omega^+} \frac{g(Du)}{D} |f|^n (1 + |Du|^2)^{\frac{n}{2}}$$

$$D = \det a_{ij} = \frac{1}{1 + |Du|^2} \begin{pmatrix} 1 - \frac{|Du|^2}{1 + |Du|^2} & 0 \\ 0 & 1 \end{pmatrix}$$

故总是能做到到

$$\begin{pmatrix} u_1 = |Du| \\ u_i = 0 \quad i \geq 2 \end{pmatrix}$$

$$\text{Take } g(\xi) = \frac{1}{(1 + |\xi|^2)^{\frac{n}{2}+1}}$$

$$\int_0^M \frac{t^{n-1}}{(1+t^2)^{\frac{n}{2}+1}} dt \leq C_n \left(\int_{\Gamma} \| \cdot \|_{L^n(\Gamma)}^n \right)^{\frac{1}{n}} < \int_0^\infty \frac{t^{n-1}}{(1+t^2)^{\frac{n}{2}+1}} dt$$

$\Rightarrow M$ 有界 (在 f 不太大时)

Thm Let $u \geq 0$, $B_r(0) \subseteq \Omega$ be a solution to (*)

Suppose $\|f\|_{C^{0,1}(\Omega)} \leq C_0$. Then $|Du(0)| \leq \exp(C_1 + C_2 \frac{M^2}{r^2})$ $M = \sup_{\Omega} u$
 $\swarrow$ n.M.C. $\nwarrow$ n.C.

Proof: X.-J. Wang (1998 Math Z)

Apply MP to $G = \eta(x) \varphi(u) \log u$, $\eta = (1 - \frac{|x|^2}{r^2})_+$, $\varphi = 1 - \frac{u}{M}$

□

lect #3 Schauder estimate

$$\Delta u = f \text{ in } \Omega.$$

Thm Let $u \in C^{2,\alpha}(\Omega)$ be a solution to $\Delta u = f$ in B_1 $d = |x-y|$

Then $\forall x, y \in B_{1/2}$

(X.J. Wang) $|D^2 u(x) - D^2 u(y)| \leq C_n \left[d \|u\|_{L^\infty(B_1)} + \int_0^d \frac{\omega(r)}{r} + d \int_{c,d}^1 \frac{\omega(r)}{r^2} \right]$

2006 C_n, C_0, C_1 are universal const.

$$\omega(r) = \sup_{|x-y| \leq r} |f(x) - f(y)|$$

$$f \in C^\alpha, \omega(r) = r^\alpha, \int_0^d \frac{\omega(r)}{r} = d^\alpha$$

$$d \int_d^1 \frac{\omega(r)}{r^2} = d \int_d^1 r^{\alpha-2} = \square r^{\alpha-1} \Big|_d^1 \approx d \square + d^\alpha$$

$$|D^2 u(x) - D^2 u(y)| \leq C \left[d \|u\|_{L^\infty(B_1)} + C d^\alpha + d C + d^\alpha \right]$$

$$\frac{|D^2 u(x) - D^2 u(y)|}{|x-y|^\alpha} \leq C \left[d^{1-\alpha} \|u\|_{L^\infty} + C_1 \right]$$

$\Rightarrow [D^2 u]_\alpha$ is bounded.

$$a_{ij} u_{ij} + b_i u_i + c u = f$$

变系数

$$a_{ij}(x_0) u_{ij} = \underbrace{(a_{ij}(x_0) - a_{ij}(x))}_{\text{仿射变换}} u_{ij} + b_i u_i + c u = f$$

$$\hat{u}(y) = u\left(\frac{Ly}{x}\right) \quad \text{仿射变换}$$

$$\hat{f} \in C^\alpha \text{ if}$$

就变为

(X.J. Wang 2006)

Thm Let $f \in C^\alpha(\Omega)$ and $u \in C^{2,\alpha}(\Omega)$ be a solution to
$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$
 $\downarrow C^{2,\alpha}$
 $(\lambda \delta_{ij} \leq a_{ij} \leq \Lambda \delta_{ij}, a_{ij}, b_i, c \in C^\alpha(\Omega))$
 For $\Omega' \subset \subset \Omega$, $\exists C = C(\Omega', \Omega, \alpha, n, a_{ij}, b_i, c, \lambda, \Lambda)$
 st. $\|u\|_{C^{2,\alpha}(\Omega')} \leq C(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^\alpha(\Omega)})$

若边界值: $v = u - \varphi$,
$$\begin{cases} \mathcal{L}v = \mathcal{L}u - \mathcal{L}\varphi = \hat{f} \in C^\alpha \\ v = 0 \end{cases}$$

Application: (Continuity method)

Thm $\begin{cases} \Delta u = f \\ u = g \end{cases}$ has a unique $C^{2,\alpha}$ solution for any $f \in C^\alpha, g \in C^{2,\alpha}$

$$\Rightarrow \begin{cases} \mathcal{L}u = f \\ u = g \end{cases} \quad \dots \dots \dots$$

($C < \infty$ 唯一性?)

Idea $\mathcal{L}_t = t\mathcal{L} + (1-t)\Delta$, $I = \{t \in [0,1]\}$: $\begin{cases} \mathcal{L}_t u = f \\ u = g \end{cases} \exists! C^{2,\alpha} \text{ solution}$

前提: 全局

$|t-s| < \delta$, 可构造

Schauder estimate.

Fix t , T_t 互缩映射, $\mathcal{D} \in I$.

$$\forall f \in C^\alpha, g \in C^{2,\alpha} \left\{ \right.$$



$\Delta u = f$ (follow argument of X. J. Wang 2006)

Given $z \in B_{k+3} \setminus B_{k+4}$, $B_k = B_{\rho^k}$ $f = \frac{1}{2}$

We hope to estimate

($k \uparrow$ ball $\downarrow$)

$$|D^2 u(z) - D^2 u(0)| \leq C \left(|z| \sup_{B_1} |u| + \int_0^{|z|} \frac{\omega(r)}{r} dr + |z| \int_{|z|}^1 \frac{\omega(r)}{r^2} dr \right)$$

$$\omega(r) = \sup_{|x-y| \leq r} |f(x) - f(y)|$$

Tool: maximum principle and interior estimates of harmonic function

$$\text{Consider } \begin{cases} \Delta u_k = f(x) & \text{in } B_k \\ u_k = u & \text{on } \partial B_k \end{cases}$$

$$\begin{aligned} 1^\circ \quad w_{\pm} &= u - u_k \pm \frac{\omega(\rho^k)}{2n} (\rho^{2k} - |x|^2) \quad \text{in } B_k \\ \begin{cases} \Delta w_{\pm} &= f - f(x) \mp \omega(\rho^k) & \text{in } B_k \\ w_{\pm} &= 0 & \text{on } \partial B_k \end{cases} \quad \begin{cases} \Delta w_{+} \leq 0 & \text{in } B_k \\ w_{+} = 0 & \text{on } \partial B_k \end{cases} \end{aligned}$$

$$w_{+} \geq 0, \quad w_{-} \leq 0 \Rightarrow u_k - \frac{\omega(\rho^k)}{2n} (\rho^{2k} - |x|^2) \leq u \leq u_k + \frac{\omega(\rho^k)}{2n} (\rho^{2k} - |x|^2)$$

$$\Rightarrow \|u - u_k\|_{\infty, B_k} \leq C \rho^{2k} \omega(\rho^k)$$

$$\begin{aligned} \Rightarrow \|u_{k+1} - u_k\|_{\infty, B_{k+1}} &\leq \|u - u_k\|_{\infty, B_k} + \|u - u_{k+1}\|_{\infty, B_k} \\ &\leq C \rho^{2k} \omega(\rho^k) \end{aligned}$$

$$2^\circ \quad \text{Consider } Q(x) = u(x) + x \cdot Du(x) + \frac{1}{2} x \cdot D^2 u(x) \cdot x$$

$$\Delta(Q - u_k) = 0$$

$$\|u_k - Q\|_{\infty, B_k} \leq \frac{\|u_k - u\|_{\infty, B_k}}{C \rho^{2k} \omega(\rho^k)} + \frac{\|u - Q\|_{\infty, B_k}}{o(\rho^{2k})}$$

By interior estimates

$$|D^2 u_k(x) - D^2 u(x)| \leq \frac{C}{\rho^{2k}} (\rho^{2k} \omega(\rho^k) + o(\rho^{2k}))$$

$$\begin{aligned} |D^2 u(x)| &\leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B_{1/2}(x))} \\ &\leq \frac{C_k}{r^k} \|u\|_{\infty, B_k} \end{aligned}$$

$\longrightarrow 0$

$$|D^2 u(z) - D^2 u(0)| = \underbrace{|D^2 u_{k_0}(z) - D^2 u_{k_0}(0)|}_{I_1} + \underbrace{|D^2 u(0) - D^2 u_{k_0}(0)|}_{I_2} + \underbrace{|D^2 u_{k_0}(z) - D^2 u(z)|}_{I_3}$$

For I_2 : $I_2 \leq \sum_{j=k_0}^{\infty} |D^2 u_{j+1}(0) - D^2 u_j(0)| \leq \sum_{j \geq k_0} \frac{\rho^{2j}}{(\rho^j)^2} \omega(\rho^j)$

$\parallel$
 $D^2(u_{j+1} - u_j)(0)$
harmonic

$\leq \frac{C}{1-\rho} \sum_{j \geq k_0} \frac{\omega(\rho^j)}{\rho^{j-1}} \rho^{j-1}(1-\rho)$

$= C \sum_{j \geq k_0} \frac{\omega(\rho^j)}{\rho^{j-1}} (\rho^j - \rho^{j-1})$

$\leftarrow$

$\leq C \int_0^{C \cdot |z|} \frac{\omega(r)}{r} dr$

$\int_{\rho^j}^{\rho^{j+1}} \frac{\omega(r)}{r} dr$

For I_3 , Consider v_k $\begin{cases} \Delta v_k = f(z) & \text{in } B_k(z) \\ v_k = u & \text{on } \partial B_k(z) \end{cases}$
 (Exercises)

$$I_3 \leq \underbrace{|D^2 u_{k_0}(z) - D^2 v_{k_0}(z)|}_? + |D^2 v_{k_0}(z) - D^2 u(z)|$$

$$\Delta(u_{k_0} - v_{k_0}) = f(0) - f(z) = \Delta \left(\frac{f(0) - f(z)}{2n} (|x|^2 - \rho^{2k}) \right)$$

For I_1 , $I_1 \leq |D^2 u_{k_0-1}(z) - D^2 u_{k_0-1}(0)| + |D^2(u_{k_0} - u_{k_0-1})(z) - D^2(u_{k_0} - u_{k_0-1})(0)|$

$$\leq |D^2 u_0(z) - D^2 u_0(0)| + \sum_{j=1}^{k_0} |D^2 h_j(z) - D^2 h_j(0)| \rho^{2k_0}$$

$$\leq |z| \cdot |D^3 u_0|_{\infty, B_{\frac{1}{2}}} + C|z| \sum_{j=1}^{k_0} |D^3 h_j|_{\infty, B_{j-1}} \leq \frac{\omega(\rho^j)}{\rho^{j-1} \rho^{j-1}} \frac{\rho^{j-1}(1-\rho)}{\rho^j - \rho^{j-1}}$$

$\Rightarrow \int_{\rho^j}^{\rho^{j+1}} \frac{\omega(r)}{r^2} dr$

$$I_1 \leq |z| \cdot |D^3 u_0|_{\infty, B_{\frac{1}{2}}} + C|z| \sum_{j=1}^{k_0} \int_{\rho^j}^{\rho^{j+1}} \frac{\omega(r)}{r^2} dr$$

$\int_0^1 \frac{\omega(r)}{r^2} dr$

$$\Delta u_0 = f(0) = \Delta \left(\frac{f(0)}{2n} (|x|^2 - 1) \right)$$

$$|D^3 u_0| = \left| D^3 \left[u_0 - \left(\frac{f(0)}{2n} (|x|^2 - 1) \right) \right] \right| \xrightarrow{\text{harmonic}} \leq \|u_0 - \frac{f(0)}{2n} (|x|^2 - 1)\|_{\infty} \leq \sup_{B_1} |u_0|$$

$$\|D^2 u\|_{C^{2,\alpha}(B_{\frac{1}{2}})} \leq C (\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)})$$

$$\Delta u = f \text{ in } \Omega, \|u\|_{C^{2,\alpha}(\bar{\Omega})} \leq C_{\Omega, \Omega'} (\|u\|_{L^\infty} + \|f\|_{C^\alpha(\Omega)})$$

Thm 1. Let $\Omega \in C^{2,\alpha}$, bounded $f \in C^\alpha(\bar{\Omega})$, $\phi \in C^{2,\alpha}(\bar{\Omega})$,
 and $u \in C^{2,\alpha}(\bar{\Omega})$ be a solution of $\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases}$
 (uniformly elliptic. $+ \|a_{ij}\|_{C^\alpha}, \|b_i\|_{C^\alpha}, \|c\|_{C^\alpha} \leq \Lambda$)
 then $\exists C = C(n, \lambda, \Lambda, \Omega)$ s.t.

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leq C (\|u\|_{L^\infty(\bar{\Omega})} + \|\phi\|_{C^{2,\alpha}(\bar{\Omega})} + \|f\|_{C^\alpha(\bar{\Omega})})$$

Thm Let $\Omega, \phi, \mathcal{L}$ as above, $c \leq 0$ in Ω . Then $\forall f \in C^\alpha(\bar{\Omega})$
 $\phi \in C^{2,\alpha}(\bar{\Omega})$, $\exists! u \in C^{2,\alpha}(\bar{\Omega})$ solves

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega. \end{cases}$$

key ingredient Let $\Omega, \mathcal{L}$ be as Thm and $\phi \in C^{2,\alpha}(\bar{\Omega})$
 unique
 if $\begin{cases} \Delta u = f & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases}$ has a $\forall C^{2,\alpha}(\bar{\Omega})$ solution for all $f \in C^\alpha(\bar{\Omega})$
 then $\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases}$ also has a (unique) $C^{2,\alpha}(\bar{\Omega})$ solution for all such f and ϕ

Consider a family of operator $\mathcal{L}_t = t\mathcal{L} + (1-t)\Delta \quad t \in [0, 1]$

$$a_{ij}^t \geq \lambda' \delta_{ij} / |a_{ij}^t|_{C^\alpha}, \|b_i^t\|_{C^\alpha}, \|c^t\|_{C^\alpha} \leq \Lambda, \quad \lambda' \& \Lambda \text{ indep. of } t$$

Define $I = \{s \in [0, 1] : \begin{cases} \mathcal{L}_s w = f & \text{in } \Omega \\ w = \phi & \text{on } \partial\Omega \end{cases} \text{ has a (unique) solution for all } f \in C^\alpha(\bar{\Omega}) \text{ and } \phi \in C^{2,\alpha}(\bar{\Omega})\}$

$0 \in I$ δ (universal)



Assume $s \in I$. We expect to show, $\exists$ universal δ s.t.

if $|s-t| \leq \delta$, then $t \in I$

$$\text{if } \mathcal{L}_t u = f, \quad \mathcal{L}_s u = \mathcal{L}_s u + f - \mathcal{L}_t u = f + s \mathcal{L} u + (1-s) \Delta u - (t \mathcal{L} u + (1-t) \Delta u) \\ = f + (s-t) (\mathcal{L} - \Delta) u \iff u = \mathcal{L}_s^{-1} (f + (s-t) \Delta u) = T_t u$$

Define $T_t : \mathcal{X} = \{u \in C^{2,\alpha}(\bar{\Omega}), u|_{\partial\Omega} = 0\} \longrightarrow \mathcal{X}$

$u \mapsto T_t u = \mathcal{L}_s^{-1} (f + (s-t) (\mathcal{L} - \Delta) u)$

Banach space.

$\mathcal{L}_s^{-1} \psi$ is the solution of $\begin{cases} \mathcal{L}_s w = \psi & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \quad (\because C \in 0, \text{ resp. } \lambda = 1)$

We show that T_t is a contraction maps (?)

$$\begin{aligned} \|T_t u - T_t v\|_{C^{2,\alpha}} &= \|\mathcal{L}_s^{-1} ((s-t) (\mathcal{L} - \Delta) (u-v))\|_{C^{2,\alpha}} \\ &= |s-t| \|\mathcal{L}_s^{-1} ((\mathcal{L} - \Delta) (u-v))\|_{C^{2,\alpha}} \\ &\leq C |s-t| \|(\mathcal{L} - \Delta) (u-v)\|_{C^\alpha} \quad (\text{maximal principle \& Schauder}) \\ &\leq C' |s-t| \|u-v\|_{C^{2,\alpha}} \\ &\leq \frac{1}{2} \end{aligned}$$

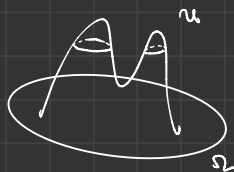
$$E(u) = \int_{\Omega} F(\nabla u), \quad F(\eta) = \sqrt{1 + |\eta|^2}$$

$$0 = \frac{d}{dt} E(u + t\varphi) = \int_{\Omega} \frac{d}{dt} F(\nabla(u + t\varphi)) \\ = \int_{\Omega} F_{\eta_i} \varphi_i = - \int_{\Omega} \partial_i [F_{\eta_i}(\nabla u)] \varphi$$

$$\Rightarrow 0 = \partial_i [F_{\eta_i}(\nabla u)] = F_{\eta_i \eta_j} \partial_j u, \quad \lambda \delta_{ij} \in F_{\eta_i \eta_j}(\nabla u) \leq \Lambda \delta_{ij}$$

Take $\text{Arg}(u) = F_{\eta_i \eta_j}(\nabla u) \in C^\alpha \xrightarrow{\text{Schauder}} u \in C^{2,\alpha} \quad u \in C^1(\Omega)$

$u \in C^{1,\alpha} \xrightarrow{\text{bootstraping}} u \in C^\infty$



$$\partial_i (F_{kij} \partial_j u) = 0$$

$$\Omega \rightarrow u \text{ satisfies } \partial_i (a_{ij} \partial_j u) = 0, \quad a_{ij} = F_{kij}(Du)$$

$$\lambda \delta_{ij} \leq a_{ij} \leq \Lambda \delta_{ij} \Rightarrow \underline{u} \in C^\alpha$$

De Giorgi - Moser (1960s)

Thm Let $u \in H^1(B_1)$ be a solution of $\partial_i (a_{ij} \partial_j u) = f$

(i) $\lambda \delta_{ij} \leq a_{ij} \leq \Lambda \delta_{ij}$, (ii) $f \in L^q(B_1)$, $q > \frac{n}{2}$. Then

$$|u|_{C^\alpha(B_{\frac{1}{2}})} \leq C \left(|u|_{L^2(B_1)} + |f|_{L^q(B_1)} \right)$$

$\downarrow$
n. l. n. q

Local boundedness (From L^2 to L^∞ bound)

prop 1 If $\partial_i (a_{ij} \partial_j u) \geq f$ in B_1 , $f \in L^q(B_1)$, $q > \frac{n}{2}$. Then

$$\sup_{B_{\frac{1}{2}}} u^+ \leq C \left(|u^+|_{L^2(B_1)} + |f|_{L^q(B_1)} \right)$$

Def $\partial_i (a_{ij} \partial_j u) \geq f$ in B_1 means $\forall \varphi \in H_0^1(B_1)$ and $\varphi \geq 0$. We have

$$-\int a_{ij} \varphi_i u_j = \int \varphi \partial_i (a_{ij} \partial_j u) \geq \int \varphi f$$

$$\Rightarrow \int a_{ij} \varphi_i u_j \leq - \int \varphi f$$

Sobolev (Evans, chapter 5)

$$|w|_{L^2(\Omega)} \lesssim |w|_{L^2(\Omega)} \leq C_{\varepsilon^*} |Dw|_{L^2(B_1)}$$

$$(W^{k,p} \hookrightarrow L^{\frac{np}{n-kp}})$$

$$2^* = \begin{cases} \frac{2n}{n-2} \quad (>2) \quad n \geq 3 \\ \text{any } p > 2 \quad n=2 \end{cases}$$

Energy estimates

Lemma If $\forall \varphi \in H_0^1(B_1)$ $\partial_i (a_{ij} \partial_j u) \geq f \in L^q(B_1)$, $q > \frac{n}{2}$. then $\forall \eta \in H_0^1(B_1)$

$$\int_{B_1} |D(\eta u)|^2 \leq C \sup_{B_1} |D\eta|^2 \int_{\text{supp } \eta} u^2 + C \sup_{B_1} \eta^2 |f|_{L^q}^2 |\text{supp } (\eta u)|^{1-\frac{1}{q}}$$

$\nearrow$ cut-off func

Proof: Take $g = u\eta^2$ in Definition

$$\int_{B_1} a_{ij} \partial_i (u\eta^2) u_j \leq - \int_{B_1} f u \eta^2$$

$$\text{L.h.s.} = \int_{B_1} a_{ij} (\eta \partial_i (u\eta) + u \eta \partial_i \eta) u_j = \int_{B_1} a_{ij} \partial_i (u\eta) [\partial_j (u\eta) - u \partial_j \eta]$$

$$\geq \lambda \int_{B_1} |D(u\eta)|^2 - \int_{B_1} a_{ij} (u \eta \partial_i \eta) u_j + \int_{B_1} a_{ij} u \eta \eta_i u_j$$

$$\geq \lambda \int_{B_1} |D(u\eta)|^2 - \lambda \sup_{B_1} |D\eta|^2 \int_{\text{supp } \eta} u^2$$

$$\begin{aligned} \text{r.h.s.} &\leq \int |f| |u\eta| |\eta| \leq \sup_{B_1} |\eta| \|f\|_{L^q} \|u\eta\|_{L^{q^*}} \\ &\stackrel{\text{Holder}}{\leq} \frac{1}{\delta} \sup_{B_1} \eta^2 \|f\|_{L^q}^2 |\text{supp}(u\eta)|^{2-\frac{2}{q}-\frac{2}{q^*}} + \delta \|u\eta\|_{L^{q^*}}^2 \\ &\leq \frac{1}{\delta} \sup_{B_1} \eta^2 \|f\|_{L^q}^2 |\text{supp}(u\eta)|^{2-\frac{2}{q}-\frac{2}{q^*}} + \delta \|u\eta\|_{L^{q^*}}^2 \leq C \int |D(u\eta)|^2 \\ \delta C &\leq \frac{\lambda}{2} \quad (\text{取 } \delta \text{ 足够小}) \end{aligned}$$

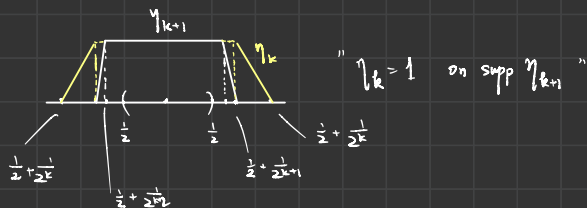
$$2 - \frac{2}{q} - \frac{2}{q^*} = 1 - \frac{1}{q} + \left[1 - \frac{1}{q} - \frac{2}{q^*}\right]_{>0}$$

□

Remark: if $\partial_i (a_{ij} \partial_j u) \geq f$, then $\partial_i (a_{ij} \partial_j u) \geq -|f|$.

Proof of Prop 1

1° cut-offs: $\eta_k \geq \eta_{k+1} \rightarrow 1_{B_{\frac{1}{2}}}$ $\eta_k = \begin{cases} 1 & B_{\frac{1}{2} + \frac{1}{2^{k+1}}} \\ |D\eta_k| \kappa 2^k & B_{\frac{1}{2} + \frac{1}{2^k}} \setminus B_{\frac{1}{2} + \frac{1}{2^{k+1}}} \\ 0 & \text{outside } B_{\frac{1}{2} + \frac{1}{2^k}} \end{cases}$

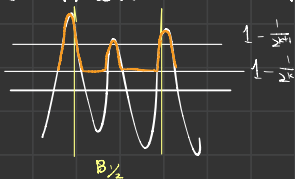


$$\hat{u} = \frac{u}{\varepsilon + \delta_0^{-1} (\|u^*\|_{L^2} + \|f\|_{L^q})}, \quad \|\hat{u}\|_{L^2} = \frac{\|u^*\|_{L^2}}{\varepsilon + \delta_0^{-1} (\|u^*\|_{L^2} + \|f\|_{L^q})} \leq \delta_0$$

$$\int \hat{u} = \frac{\int f}{\varepsilon + \delta_0^{-1} (\|u^*\|_{L^2} + \|f\|_{L^q})} =: \hat{f}, \quad \|\hat{f}\|_{L^q} \leq \delta_0$$

转化为无穷级数. $\exists$ universal δ_0 s.t. $\|\hat{u}\|_{L^2}, \|\hat{f}\|_{L^1} \leq \delta_0 \Rightarrow \sup_{B_{1/2}} \hat{u}^+ \leq 1$

2° Tractions.



$$u_k = [u - (1 - \frac{1}{2^k})]^+$$

Consider $\eta_k u_k$

$$(i) \eta_k u_k \geq \eta_{k+1} u_{k+1} \longrightarrow [u-1]^+ 1_{B_{1/2}}$$

$$(ii) \eta_k u_k > \frac{1}{2^{k+1}} \quad \text{on} \quad \text{supp}(\eta_{k+1} u_{k+1})$$

3° Iteration. Consider $V_k = \int \eta_k^2 u_k^2$

$$V_k = \left| \eta_k u_k \right|_{L^2}^2 \stackrel{\text{Holder}}{\leq} \left| \eta_k u_k \right|_{L^{2^*}}^2 \left| \text{supp} \eta_k u_k \right|^{1 - \frac{2}{2^*}}$$

$$\stackrel{\text{Sobolev}}{\leq} C \int |\nabla(\eta_k u_k)|^2 \left| \text{supp} \eta_k u_k \right|^{1 - \frac{2}{2^*}}$$

$$\text{Energy estimates} \leq C 2^{2k} \int_{\text{supp} \eta_k} u_k^2 \left| \text{supp} \eta_k u_k \right|^{1 - \frac{2}{2^*}} + C \|f\|_{L^1}^2 \left| \text{supp} \eta_k u_k \right|^{2 - \frac{1}{q} - \frac{2}{2^*}}$$

$$V_{k+1} \leq C 2^{2k} \underbrace{\int \eta_k^2 u_k^2}_{V_k} \underbrace{\left| \text{supp} \eta_{k+1} u_{k+1} \right|^{1 - \frac{2}{2^*}}}_{2^{-\frac{1}{q}}} + C \|f\|_{L^1}^2 \underbrace{\left| \text{supp} \eta_{k+1} u_{k+1} \right|^{2 - \frac{1}{q} - \frac{2}{2^*}}}_{2^{-\frac{1}{q} - \frac{2}{2^*}}}$$

$$V_k \geq \int_{\text{supp}(\eta_{k+1} u_{k+1})} \eta_k^2 u_k^2 \geq 2^{-(k+1)} \left| \text{supp}(\eta_{k+1} u_{k+1}) \right| \longrightarrow \underline{| \sim | \leq 2^{k+1} V_k}$$

$$V_{k+1} \leq C 2^{2k} V_k \underbrace{\left| \text{supp} \eta_{k+1} u_{k+1} \right|^{2 - \frac{1}{q} - \frac{2}{2^*}}}_{2^{-\frac{1}{q} - \frac{2}{2^*}}} + C (2^{2k} V_k)^{2 - \frac{1}{q} - \frac{2}{2^*}} \stackrel{1 + \varepsilon}{\leq} 1 + \varepsilon \quad \text{if } 2 = 1 + \frac{1}{q} - \frac{2}{2^*} > 0$$

$$\Rightarrow V_{k+1} \leq C (2^{2k} V_k)^{2 + \varepsilon} + C (2^{2k} V_k)^{1 + \varepsilon}$$

$$\leq C 2^{\frac{2k(1+\varepsilon)}{a}} V_k^{1+\varepsilon} \quad \text{if } a = 2^{2(1+\varepsilon)}$$

$$V_{k+1} \leq C a^k V_k^{1+\varepsilon} \leq C a^k (C a^{k-1} V_{k-1}^{1+\varepsilon})^{1+\varepsilon}$$

$$= C^{1+(1+\varepsilon)} a^{k+(k-1)(1+\varepsilon)} V_{k-1}^{(1+\varepsilon)^2}$$

$$\leq C^{\sum_{j=0}^k (1+\varepsilon)^j} a^{\sum_{j=0}^k (1+\varepsilon)^{k-j}} V_0^{(1+\varepsilon)^{k+1}}$$

$$= \left[C^{\frac{k}{\sum_{j=0}^k (1+\varepsilon)^j} \frac{j-(k-1)}{<0}} a^{\frac{k}{\sum_{j=0}^k (1+\varepsilon)^{k-j} \frac{j-1}{<0}} \right] (1+\varepsilon)^{k+1} \delta_0$$

$\nexists \delta_0(C, a, \varepsilon)$

足够小.

$$\leq \left(\frac{1}{2}\right)^{(1+\varepsilon)^{k+1}} \longrightarrow 0$$

convergence.

$\therefore$ limits are bounded.

$$\begin{aligned} V_0 &= \int \eta_0^2 u^2 \\ &\leq \|u\|_{L^2(B_1)}^2 \\ &\leq \delta_0. \end{aligned}$$

$$\eta_k u_k \longrightarrow (u-1)^+ 1_{B_{1/2}} \quad \sup_{B_{1/2}} u^+ \leq 1$$

Thm $\mathcal{L}u \triangleq \partial_i (a_{ij} \partial_j u) = f \in L^q(B_1)$, $q > n/2$, $u \in C^\alpha(B_1)$, $\alpha > n/2$

$$u \in C^\alpha(B_{1/2}), \forall x, y \in B_{1/2}, |u(x) - u(y)| \leq C |x - y|^\alpha (\|u\|_{L^2(B_1)} + \|f\|_{L^q(B_1)})$$

We show $|u(x) - u(0)| \leq C |x|^\alpha (\|u\|_{L^2(B_1)} + \|f\|_{L^q(B_1)})$

Prop (Oscillation, decay) $\exists$ universal $\sigma \in (0,1)$ and $C > 0$, s.t.

$$\text{Osc}_{B_{1/2}} u \leq \sigma \text{Osc}_{B_1} u + C \|f\|_{L^q(B_1)}$$

proof of Hölder estimates $B_{1/2^k}$

Consider $\hat{u}|_{B_1} = u|_{B_{1/2^k}}$, $y \in B_1$.

$$\mathcal{L} \hat{u}|_{B_1} = 4^k f(y/2^k) \triangleq \hat{f}(y) \text{ in } B_1$$

Apply Oscillation, decay to $\hat{u}$ in B_1 .

$$\text{Osc}_{B_{1/2}} \hat{u} \leq \sigma \text{Osc}_{B_1} \hat{u} + C \|\hat{f}\|_{L^q(B_1)} = \left(4^{-kq} \int f^q(y/2^k) dy \right)^{1/q}$$

$$\Rightarrow \text{Osc}_{B_{1/2}} u \leq \sigma \text{Osc}_{B_{1/2}} u + C a^k \|f\|_{L^q(B_1)} = 4^{-k} \left(2^{kq} \int f^q(y/2^k) dy / 2^q \right)^{1/q}$$

$$+ C a^k \|f\|_{L^q(B_1)} \leq 4^{-k} \cdot 4^{k\alpha/2} \|f\|_{L^q(B_1)}$$

$$\leq \sigma (\text{Osc}_{B_{1/2^{k-1}}} u + C a^{k-1} \|f\|_{L^q}) + C a^k \|f\|_{L^q}$$

$$= \sigma^2 \text{Osc}_{B_{1/2^{k-1}}} u + C \|f\|_{L^q} (\sigma a^{k-1} + a^k)$$

$$\leq \sigma^k \text{Osc}_{B_{1/2}} u + C \|f\|_{L^q} \left(\sum_{j=0}^{k-1} \sigma^j a^{k-j} \right)$$

$$4^{k(1-\frac{\alpha}{2})} = 2 > 0$$

$$a = 4^{-\alpha/2} < 1$$

$$\text{Let } \tau = \max \{ \sigma, a \}$$

$$\leq \tau^k C (\|u\|_{L^2(B_1)} + \|f\|_{L^q(B_1)}) + C \|f\|_{L^q} \tau^k = |x|^2 (-\log |x|)$$

Suppose $x \in B_{1/2^k} \setminus B_{1/2^{k+1}}$, $|x| \approx 2^{-k}$, k large

$$\tau^k = 2^{\log_2 \tau^k} = 2^{-k |\log_2 \tau|} \approx |x|^{\log_2 \tau}$$

Proof of oscillation decay

Consider $\hat{u}(x) = \frac{\partial u(x)}{\partial \nu_{B_1(x)}} \quad \tilde{u}(x) = \hat{u}(x) - \frac{1}{2} \left(\sup_{B_1} \hat{u} + \inf_{B_1} \hat{u} \right)$

$\implies \text{osc}_{B_1} \tilde{u} = 2 \quad \sup_{B_1} \tilde{u} = 1 \quad \inf_{B_1} \tilde{u} = -1$

$\mathcal{L} \tilde{u} = \frac{2f}{\text{osc}_{B_1} u} = \tilde{f}$

Let δ be a universal const (to be determined)

(i) $\|f\|_{L^q} > \delta \quad (2\|f\|_{L^q} > \text{osc}_{B_{1/2}} u, \quad \text{osc}_{B_{1/2}} u = \frac{1}{2} \text{osc}_{B_1} u + \frac{1}{2} \text{osc}_{B_1} u \leq \frac{1}{5} \|f\|_{L^q})$

(ii) $\|f\|_{L^q} \leq \delta \quad \text{either } |\{\tilde{u}^+ = 0\} \cap B_{3/4}| \geq \frac{1}{2} |B_{3/4}| \xrightarrow{\text{lem}} \tilde{u}^+ \leq 1-\delta$
 or $|\{\tilde{u}^- = 0\} \cap B_{3/4}| \geq \frac{1}{2} |B_{3/4}| \quad \frac{2 \text{osc}_{B_{1/2}} u}{\text{osc}_{B_1} u} = \text{osc}_{B_{1/2}} \tilde{u} \leq \frac{(2-\delta)/2}{\delta} = \frac{1-\delta}{\delta}$

Lemma 1 Let $0 \leq u \leq 1, \mathcal{L}u = f$ Assume

$|\{u=0\} \cap B_{3/4}| \geq (1-\mu) |B_{3/4}| \quad \text{for some } \mu \in (0, 1)$

Then $\exists \delta = \delta_\mu \quad \delta = \delta_\mu$ s.t. $\sup_{B_{1/2}} u \leq 1 - \delta_\mu$ provided $\|f\|_{L^q} \leq \delta_\mu$

Recall: $\sup_{B_{1/2}} u \leq C \left(\|u\|_{L^2(B_{3/4})} + \|f\|_{L^q(B_1)} \right) \quad \hookrightarrow, \quad |\eta_k u_k|_2 \leq |\sup \eta_k u_k|_2^{\frac{1}{2-\frac{1}{2^k}}}$

Consider $u_k = \left[u - \left(1 - \frac{1}{2^k}\right) \right]_+$

$\hat{u}_k = 2^k u_k, \quad 0 \leq \hat{u}_k \leq 1$

$|\eta_k u_k|_{2^*} \leq |D \eta_k u_k|_2 \leq |\eta_k u_k|_2$

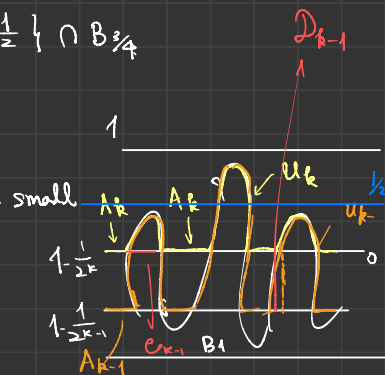
Define $A_k = \{\hat{u}_k = 0\} \cap B_{3/4}$

$C_k = \{\hat{u}_k \geq \frac{1}{2}\} \cap B_{3/4}$

$D_k = \{0 < \hat{u}_k < \frac{1}{2}\} \cap B_{3/4}$

We aim to show $\exists k_0$ (finite) s.t.

$|A_{k_0}| \approx |B_{3/4}| \implies \sup_{\substack{\text{estimate} \\ \text{to } \hat{u}_k \text{ in } B_{3/4}}} \hat{u}_{k_0} \leq \epsilon_0$



Transform this back to u . we get the conclusion

Pf: Lem 1

$$\exists k=k_0 \text{ s.t. } |A_k| \geq (1-\delta_0) |B_{3/4}| \text{ for a } \delta_0$$

$$\sup_{B_{1/2}} \hat{u}_{k_0} \leq \frac{C \delta_0 |B_{3/4}|^{\frac{1}{2}}}{< \frac{1}{4} \text{ (take } \delta_0 \text{ small)}} + \frac{4^{k_0} \|f\|_{L^q(B_1)}}{< \frac{1}{4}} < \frac{1}{2}$$

$$\Rightarrow \sup_{B_{1/2}} u \leq 1 - \frac{1}{2^{k_0+1}}$$

Assume NOT. $(1-\mu) |B_{3/4}| \leq |A_0| \leq |A_k| \leq (1-\delta_0) |B_{3/4}| \quad \forall k$

Lemma 2 Let $w \in H^1(B_r)$, $0 \leq w \leq c$ For $c > 0$. Let

$$A = \{w=0\}, \quad C = \{w=c\}, \quad D = \{0 < w < c\}$$

$$\text{Then } \exists C_n > 0, \text{ s.t. } |D| \int_{B_r} |Dw|^2 \geq \frac{C^2 c_n}{r^{2n}} |C|^{2-\frac{2}{n}} |A|^2$$

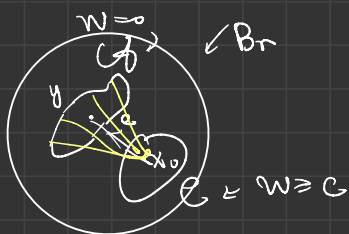
Apply Lemma 2 to u_k, C_k, D_k

$$|D_k| \geq C_\mu |C_k|^{2-\frac{2}{n}} / \boxed{\int_{B_{3/4}} |D \hat{u}_k|^2} \rightsquigarrow \left(1 + 4^k \|f\|_{L^q}^2\right)$$

$$\begin{aligned} \Rightarrow |u_k| &= |u_{k-1}| + |D_{k-1}| \geq \frac{C_\mu \delta_0^{2-\frac{2}{n}}}{1 + 4^k \|f\|_{L^q}^2} \\ &\geq |u_0| + \sum_{j=0}^{k-1} \boxed{|D_j|} \quad |u_k| + |C_{k-1}| = |B_{3/4}| \\ &\geq (1-\mu) |B_{3/4}| \quad \geq \delta_0 |B_{3/4}| \end{aligned}$$

□

Proof of Lem 2



$$C = W(x_0) - W(y)$$

$$= - \int_0^1 \dot{\gamma}_t \frac{d}{dt} W(x_0 - t \dot{\gamma}_t) dt$$

$$\leq \int_0^1 |DW(x_0 - t \dot{\gamma}_t)| dt$$

$$\text{Define } \Theta_{x_0, \mathcal{A}} = \{ e \in S^{n-1} \mid e = \frac{y - x_0}{|y - x_0|}, y \in \mathcal{A} \}$$

$$C |\Theta_{x_0, \mathcal{A}}| \leq C \int_{\Theta_{x_0, \mathcal{A}}} \int_0^1 |DW(x_0 - t e)| dt de$$

$$\frac{C_n C |\mathcal{A}|}{r^n} \leq C \int_D \frac{|DW(z)|}{|z - x_0|^{n-1}} dz$$

$$|\mathcal{A}| \geq \int_{\Theta_{x_0, \mathcal{A}}} \int_0^{2r} t^{n-1} dt de = C r^n |\Theta_{x_0, \mathcal{A}}|$$

$$\Rightarrow |\mathcal{A}| \leq \frac{C_n r^n}{C} \int_D \frac{|DW(z)|}{|z - x_0|^{n-1}} dz$$

$$|\mathcal{A}| |e| \leq \frac{C_n r^n}{C} \int_D \int_e \frac{|DW(z)|}{|z - x_0|^{n-1}} dx_0 dz \leq \frac{C_n r^n}{C} \int_D |DW(z)|$$

$$\text{Take } B_p(z) \text{ s.t. } |B_p(z)| = |e|$$

$$\Rightarrow |\mathcal{A}| |e|^{1 - \frac{1}{p}} \leq \frac{C_n r^n}{C} \int_D |DW| dz$$

$$\leq \frac{C_n r^n}{C} \left(\int_D |DW|^2 \right)^{\frac{1}{2}} |D|^{\frac{1}{2}}$$

□

$$\int_{B_p(z)} \frac{dx_0}{|z - x_0|^{n-1}} dz$$

$\downarrow$
 p^{n-1}

作了些更正



(附李老师帅照一张，doge)

Lemma Let $w \in H^1(B_r)$ $0 \leq w \leq c$ for some $c > 0$

Let $A = \{w = 0\}$, $C_0 = \{w = c\}$, $D = \{0 < w < c\}$

Then $|D| \int_{B_r} |Dw|^2 \geq \frac{c^2}{r^{n-2}} |C_0| |A|^2$

$$c = w(x_0) - w(x_1) = \int_{x_0}^{x_1} |Dw| dx$$

$$D_{\text{osc}} A = \{x \in \mathbb{R}^n : c = \frac{r^{n-2}}{r^{n-2}}\}$$

$$c |D_{\text{osc}} A| \leq c \int_{D_{\text{osc}} A} |Dw| dx$$

$$\frac{|A|^2}{r^{n-2}} \leq c \int_{D_{\text{osc}} A} |Dw| dx$$

$$Dw = \frac{1}{r} \left(u - \left(1 - \frac{x^2}{r^2}\right) \right)$$

$$A_k = \{x_k = 0\} \cap B_{\frac{r}{2}} = \{x_k = 0\}$$

$$C_k = \{x_k = \frac{1}{2}\} \cap B_{\frac{r}{2}} = \{x_k = \frac{1}{2}\}$$

$$D_k = \{0 < x_k < \frac{1}{2}\}$$

Lecl # 5 Krylov-Safonov's Harnack inequality

Thm Let $u \in C(B_1)$. $Lu \triangleq a_{ij} u_{ij} = f \in C(B_1)$, bounded.

Suppose $u \geq 0$. Then $\sup_{B_{1/2}} u \leq C \left(\inf_{B_{1/2}} u + \|f\|_{L^1(B_1)} \right)$

Corollary. $u \in C^\alpha(B_{1/2})$, $\forall x, y \in B_{1/2}$. $|u(x) - u(y)| \leq C |x - y|^\alpha (\|u\|_{L^\alpha} + \|f\|_{L^1})$

Proof. Consider $u - \inf_{B_1} u / \sup_{B_1} u - u$

$$\sup_{B_{1/2}} u - \inf_{B_1} u \leq C \left(\inf_{B_{1/2}} u - \inf_{B_1} u + \|f\|_{L^1(B_1)} \right)$$

$$\sup_{B_1} u - \inf_{B_{1/2}} u \leq C \left(\sup_{B_1} u - \sup_{B_{1/2}} u + \|f\|_{L^1(B_1)} \right)$$

$$\text{osc}_{B_1} u + \text{osc}_{B_{1/2}} u \leq C \left(\text{osc}_{B_1} u - \text{osc}_{B_{1/2}} u + 2\|f\|_{L^1(B_1)} \right)$$

$$\Rightarrow \text{osc}_{B_{1/2}} u \leq \frac{C-1}{C+1} \text{osc}_{B_1} u + \frac{2C}{C+1} \|f\|_{L^1}$$

$\sigma \in (0, 1)$

iteration ...

□

$$\hat{u} = \frac{\sup_{B_{1/2}} u}{\inf_{B_{1/2}} u + \delta^{-1} \|f\|_{L^1}} \leq K. \quad L \hat{u} = \frac{f}{\inf \hat{u}} = \hat{f}. \quad \|\hat{f}\|_{L^1} \leq \delta$$

Thm Let $u \in C(\Omega)$, $u \geq 0$. $\inf_{B_{1/2}} u \leq 1$. $Lu = f \in C(B_1)$ bounded.

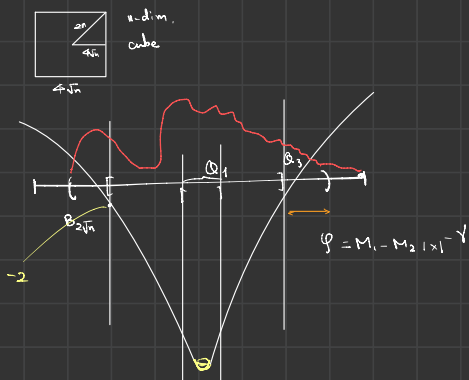
Then $\exists$ universal δ_0, K_0 s.t. $\sup_{B_{1/2}} u \leq K_0$ provided $\|f\|_{L^\infty} \leq \delta_0$.

Thm' Let $u \geq 0$ in $Q_{4\sqrt{n}}$, $\inf_{Q_{1/4}} u \leq 1$. $Lu = f \in C(Q_{4\sqrt{n}})$ bounded.

Then $\exists$ universal $\delta_0, K_0 > 0$ s.t. $\sup_{Q_{1/4}} u \leq K_0$ provided $\|f\|_{L^\infty} \leq \delta_0$.

Def. We say $u \in C(\Omega)$ is a viscosity sub/super-solution denoted by $Lu \geq f$ ($\leq$) means $\forall x_0 \in \Omega, \forall \varphi \in C^2(\Omega)$ if u, φ attain local maximum/minimum at x_0 then $L\varphi(x_0) \geq f(x_0)$ ($\leq$) / Grondie - Ishii - Lions.

(粘滞解: 唯一性, 正则性)



$$L\varphi \leq cg \quad 0 \leq g \leq 1, \quad \text{supp } g \subseteq Q_1$$

$$\varphi \leq -2 \quad \text{in } Q_3$$

$$\varphi \geq -M.$$

$$\varphi \geq 0 \quad \text{in } \mathbb{R}^n \setminus B_{2\sqrt{n}}$$

Consider $u + \varphi \mid \inf_{Q_3} (u + \varphi) \leq \inf_{Q_{1/4}} u - 2 \leq -1$

ABP - estimates
? $\downarrow$ $-(u + \varphi)$

$$Lu \geq f, \sup_{\Omega} v \leq \sup_{\partial\Omega} v^+ + C \left(\int_{\Gamma_n} |f|^n \right)^{1/n} \Rightarrow 1 \leq c \left[\int_{\Gamma_n} |f + c_0|^n \right]^{1/n} = C \|f\|_{L^n} + \left| \{u + \varphi \leq 0\} \cap Q_1 \right|^{1/n}$$

$$\Rightarrow \exists \mu_0 = \mu_0(\delta_0) \quad |\{u \leq M_0\} \cap Q_1| \geq \mu_0 |Q_1| \quad k=1, n.$$

prop. $u \geq 0$ in $Q_{4\sqrt{n}}$, $\inf_{Q_{1/4}} u \leq 1$. $Lu = f$ in $Q_{4\sqrt{n}}$, $\exists$ universal δ_0, μ_0, M_0 .

s.t. if $\|f\|_{L^n(Q_{4\sqrt{n}})} \leq \delta_0$, then $|\{u > M_0\} \cap Q_1| \leq (1 - \mu_0)^k |Q_1| \quad \forall k$

$$\Rightarrow |\{u > t\} \cap Q_1| \leq CA^{-2\alpha} \text{ (weak } L^{2\alpha}\text{-estimate)} - |k| |\log(-t_0)|$$

Proof $k=1$ ✓. By induction argument $k-1$ ✓ $|\{u > M_0^{k-1}\} \cap Q_1| \leq (1-\mu_0)^{k-1} |Q_1|$

We next show it holds for $\underline{k}$

Consider $A = \{u > M_0^k\} \cap Q_1$. $B = \{u > M_0^{k-1}\} \cap Q_1$

Lemma (CZ decomposition) $A, B \subseteq Q_1$ satisfies two conditions

(i) $|A| \leq \theta$ (ii) $\exists \{A \cap Q^i\} \rightarrow \theta |Q^i|$, then $\tilde{Q} \subseteq B$

Then $|A| \leq \theta |B|$

Proof



不相交子集

① $\{Q^i\}$, $|A \cap Q^i| > \theta |Q^i|$

② $\{\tilde{Q}^j\}$ $\tilde{Q}^j$ is a subset of Q^i
↑
disjoint

$A \subseteq \bigcup Q^i$ (except a null set)

$$|A| = |A \cap (\bigcup Q^i)| = |\bigcup (A \cap Q^i)| \leq \sum_i |A \cap Q^i|$$

$$\stackrel{\text{does not have } Q^i}{> k_1} \leq \theta \sum_i |\tilde{Q}^j| \leq \theta B. \quad \square$$

We verify
only need to

$$\theta > 1 - \mu_0$$

$$(i) |A| = |\{u > M_0^k\} \cap Q_1| \leq 1 - \mu_0 = \theta \quad \checkmark$$

(ii) $\exists Q_{2^{-i}}(x_0) \subseteq Q_1$ st. $|A \cap Q_{2^{-i}}(x_0)| > (1 - \mu_0) |Q_{2^{-i}}|$, we expect to

$$\text{show } \tilde{Q}_{2^{-i}}(x_0) = Q_{2^{-(k-1)}}(x_0) \subseteq B$$

By contradiction argument, If NOT

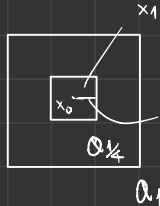
$$\exists \tilde{x} \in \tilde{Q}_{2^{-i}}(x_0) \text{ s.t. } u(\tilde{x}) \leq M_0^{k-1}$$

Consider $x = 2^i(y - x_0)$ $y \in Q_{4\sqrt{n}}$ ($\Rightarrow x \in Q_{2^{-i} \cdot 4\sqrt{n}}(x_0)$)

$$\tilde{u}(y) = \frac{u(x(y))}{M_0^{k-1}} \text{ in } Q_{4\sqrt{n}}$$

- $\tilde{u}(y) \geq 0$;
- $\inf_{Q_3} \tilde{u} \leq \frac{1}{\mu_0^{k_1}} \inf_{Q_{2^{-i}}} u = 1$; $\xrightarrow[\tilde{u} + \varphi]{ABP} |\{\tilde{u} = \mu_0\} \cap Q_1| \leq \mu_0 |Q_1|$
- $L\tilde{u} \leq \frac{f(x,y)}{\mu_0^{k_1}} = \hat{f}(y)$, $\|\hat{f}\|_{L^\infty} \leq \delta_0$. $\Rightarrow |\{\tilde{u} \leq \mu_0^{k_1}\} \cap Q_{2^{-i}}| \geq \mu |Q_{2^{-i}}|$
 $\downarrow$
 $|\{\tilde{u} > \mu_0^{k_1}\} \cap Q_{2^{-i}}| \leq (1-\mu) |Q_{2^{-i}}|$
 矛盾!

Recall : $u \geq 0$ in Q_{4^n} , $\inf_{Q_{1/4}} u \leq 1$. $L u = f$, $\exists \delta_0, k_0$ s.t. $\|f\|_{L^\infty} \leq \delta_0 \Rightarrow \sup_{Q_{1/4}} u \leq k_0$

Proof :  x_1 $u(x_1) = (1+\gamma_0) k_0$
 Q_1 $Q_{1/4}$ $Q_{1/16}$ $Q_{1/64}$ $Q_{1/256}$ $Q_{1/1024}$ $Q_{1/4096}$ $Q_{1/16384}$ $Q_{1/65536}$ $Q_{1/262144}$ $Q_{1/1048576}$ $Q_{1/4194304}$ $Q_{1/16777216}$ $Q_{1/67108864}$ $Q_{1/268435936}$ $Q_{1/1073743744}$ $Q_{1/4295075072}$ $Q_{1/17180300288}$ $Q_{1/68721201152}$ $Q_{1/274884804608}$ $Q_{1/1099539218432}$ $Q_{1/4398156873728}$ $Q_{1/17592627494912}$ $Q_{1/70370511979648}$ $Q_{1/281482047918592}$ $Q_{1/1125928191674368}$ $Q_{1/4503712766697472}$ $Q_{1/18014851066789888}$ $Q_{1/72059404267159552}$ $Q_{1/288237617068638208}$ $Q_{1/1152950468274552832}$ $Q_{1/4611801873098211328}$ $Q_{1/18447207492392845312}$ $Q_{1/73788830369571381248}$ $Q_{1/295155321478285525008}$ $Q_{1/1180621285913142100032}$ $Q_{1/4722485143652568400128}$ $Q_{1/18889940574610273600512}$ $Q_{1/75559762298441094402048}$ $Q_{1/302239049193764377608192}$ $Q_{1/1208956196775057510432768}$ $Q_{1/4835824787100230041731072}$ $Q_{1/19343303148400920166924304}$ $Q_{1/77373212593603680667697216}$ $Q_{1/309492850374414722670789248}$ $Q_{1/1237971401497658890683156992}$ $Q_{1/4951885605990635562732627968}$ $Q_{1/19807542423962542250930511872}$ $Q_{1/79230169695850169003722047488}$ $Q_{1/316920678783400676014888189952}$ $Q_{1/1267682715133602704059552759808}$ $Q_{1/5070730860534410816238211039232}$ $Q_{1/20282923442137643264952844156928}$ $Q_{1/81131693768550573059811376627712}$ $Q_{1/324526775074202292239245506510848}$ $Q_{1/1298107100296809168956982026043328}$ $Q_{1/5192428401187236675827928104173248}$ $Q_{1/20769713604748946703311712416692992}$ $Q_{1/83078854418995786813246850666771968}$ $Q_{1/332315417675983147252987402667087872}$ $Q_{1/1329261670703932589011950610668351488}$ $Q_{1/5317046682815730356047802442673405952}$ $Q_{1/21268186731262921424191210770693623808}$ $Q_{1/85072746925051685696764843082774495232}$ $Q_{1/340290987700206742787059372331097980928}$ $Q_{1/1361163950800826971148237489324391923776}$ $Q_{1/5444655803203307884592949957297567695104}$ 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Assume by contradiction argument.

$$\sup_{Q_2(x_0)} u \leq (1 + \gamma_0) K_0 \Rightarrow \forall \geq 0 \text{ in } Q_{\frac{1}{4\sqrt{n}}}$$

$$\mathcal{L}v = f$$

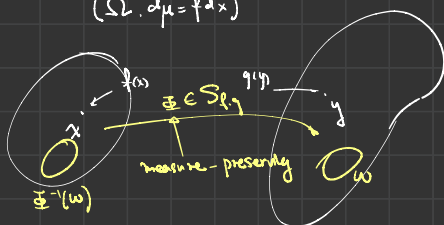
$$\text{By } L_w^2, \quad \left| \left\{ v(y) > \frac{\frac{1}{2} + \gamma}{\gamma} \right\} \cap Q_1 \right| \leq C_0 \left(1 + \frac{1}{2\gamma}\right)^{-2\alpha}$$

$$\longrightarrow \left| \left\{ u > \frac{K_0}{2} \right\} \cap Q_{\frac{1}{4\sqrt{n}}} \right| \leq C_0 \left(1 + \frac{1}{2\gamma}\right)^{-2} |Q_{\frac{1}{4\sqrt{n}}}| < \frac{1}{4} |Q_{\frac{1}{4\sqrt{n}}}|$$

$$|A| + |B| < |Q_{\frac{1}{4\sqrt{n}}}| \nrightarrow \text{!}$$

#

Let $\Omega, \Omega^* \subseteq \mathbb{R}^n$ bounded, $f \in L^1(\Omega)$, $g \in L^1(\Omega^*)$, $f \geq 0$, $g \geq 0$ (Ω^* , $d\mu^* = g dy$)
 $(\Omega, d\mu = f dx)$



$$\int_{\Omega} f = \int_{\Omega^*} g$$

(mass. balance)

Let $C: \Omega \times \Omega^* \rightarrow \mathbb{R} \longleftarrow \text{cost}$
 $\uparrow$
 cost functions

Consider $T \in S_{f,g} = \left\{ \Phi: \Omega \rightarrow \Omega^* \mid \Phi_{\#} f = g \right\}$

We say $\Phi_{\#} f = g$, if $\forall w \subseteq \Omega^*$ (Borel set)

$$\int_{\Phi^{-1}(w)} f = \int_w g$$

Monge's optimal transport problem

Find a map $T_0 \in S_{f,g}$ s.t.

$$C(T_0) = \inf_{\Phi \in S_{f,g}} C(\Phi)$$

$$\text{where } C(\Phi) = \int_{\Omega} c(x, \Phi(x)) f(x) dx \longleftarrow \text{Total cost functional}$$

History of existence.

$$C_2(x, y) = |x - y|^{1/2}$$



• 1781 G. Monge propose OT

$$C(x, y) = |x - y|$$

1985. Appell direction of transport $x \mapsto Du(x)$ (Borok Prize)

1940s. Kantorovich linear programming Nobel prize (1975)

Sudakov 1979 $n=2$

Evans-Gongbo 1991. $f, g \in C^{0,1}$, Ω, Ω^* are disjoint

Trudinger-X.J. Wang 2001 / Coflanelli-Feldman-McLennan 2002

Ambrosio 2003 fix a gap of S . $n \geq 2$

$$C(x, y) = \frac{1}{2} |x - y|^2 \quad (\Leftrightarrow C(x, y) = x \cdot y)$$

Brenier 1991 ✓ optimal $T(x) = Du(x)$, u is a convex function

$$\Phi \in S_{f, g} \quad \int_{\Phi^{-1}(w)} f = \int_w g \quad w \in \Omega^* \quad \Leftrightarrow \quad \int_{\Omega} f h \circ \Phi = \int_{\Omega^*} h g$$

$\uparrow$ $\int_{\Omega} f 1_w \circ \Phi$ $\int_{\Omega^*} 1_w g$

$\downarrow h \in C(\Omega^*)$
 $(\because 1_w \nabla \nabla \text{ continuous function with } g)$

$$C(\Phi) = \int_{\Omega} \frac{1}{2} |x - \Phi(x)|^2 f = \frac{1}{2} \int_{\Omega} |x|^2 f(x) - \int_{\Omega} x \cdot \Phi(x) f + \underbrace{\frac{1}{2} \int_{\Omega} |\Phi(x)|^2 f}_{\frac{1}{2} \int_{\Omega^*} |y|^2 g}$$

Take $h = |y|^2/2$

Thm Suppose $\Omega, \Omega^* \subset \mathbb{R}^n$ bounded, $f, g \geq 0$, $f, g \in L^1$, $\int_{\Omega} f = \int_{\Omega^*} g$

Suppose $c: \mathcal{U} \rightarrow \mathbb{R}$, $\mathcal{U} \supseteq \bar{\Omega} \times \bar{\Omega}^*$ open, $c \in C^1(\mathcal{U})$ and

$$\forall x \in \Omega, \quad D_x C(x, \cdot): \Omega^* \longrightarrow D_x C(x, \Omega^*) \subset \mathbb{R}^n$$

homeomorphism

Then $\exists T_0 \in S_{f, g}$ s.t. $C(T_0) = \inf_{\Phi \in S_{f, g}} C(\Phi)$

Moreover, $T_0(x) = [D_x C(x, \cdot)]^{-1}(-Du)$, u is c -convex function.

Kantorovich's dual functional

$$J(\varphi, \psi) = \int_{\Omega} (1-\varphi)f + \int_{\Omega^*} (1-\psi)g, \quad \mathcal{K} = \{(\varphi, \psi) \in C(\bar{\Omega}) \times C(\bar{\Omega}^*)\}$$

Kantorovich's dual problem

$$-\varphi(x) - \psi(y) \leq C(x, y)$$

$$\text{Find } (u, v) \in \mathcal{K} \text{ s.t. } J(u, v) = \sup_{\mathcal{K}} J(\varphi, \psi)$$

Def We say $u \in C(\Omega)$ is c -convex (indexed by Ω^*) if $\exists v \in C(\Omega^*)$ s.t.

$$u(x) = \sup_{y \in \Omega^*} \{ -c(x, y) - v(y) \} = v^c \leftarrow c\text{-transform}$$

$$x \cdot y - v(y) = l_y(x)$$

Thm 1 Let Ω, Ω^* f.g. c be as in Thm.

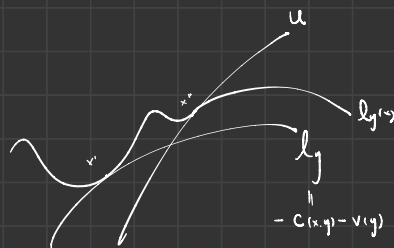
Let $u = v^c$ be a c -convex function. $T_u(x) = [D_x C(x, \cdot)]^{-1} (Du(x))$
The following are equivalent (TFAE)

(i) (u, v) are potentials of J ($J(u, v) = \sup_{\mathcal{K}} J(\varphi, \psi)$)

(ii) $T_u \in S_{f,g}$

(iii) T_u is optimal ($\mathcal{C}(T_u) = \inf_{\Phi \in S_{f,g}} \mathcal{C}(\Phi)$)

$$\text{Define } \chi_u(x) = \{y \in \bar{\Omega}^*, u(x) + v(y) = -c(x, y)\}$$



Remark

(i) $\forall x \in \Omega^*$, take $y \in \chi_u(x)$, then $u(\cdot) + c(\cdot, y)$ achieve global minimum

(ii) If u is differentiable at x , then by (i)

$$0 = Du(x) + D_x C(x, y) \Leftrightarrow y = [D_x C(x, \cdot)]^{-1} (Du(x))$$

(iii) $\forall x', x \in \Omega, y' \in \chi_u(x'), y \in \chi_u(x)$

$$\text{By (i)} \quad u(x') + c(x', y) \geq u(x) + c(x, y)$$

$$u(x) + c(x, y') \geq u(x') + c(x', y')$$

$$\frac{c(x', y') - c(x, y)}{|x - x'|} \leq \frac{u(x) - u(x')}{|x - x'|} \leq \frac{c(x', y) - c(x, y)}{|x - x'|}$$

$$\Rightarrow \|u\|_{Lip(\Omega)} \leq \sup_y \|D_x C(\cdot, y)\|_{Lip(\Omega)} \leq C$$

$$u \in C^1(\bar{\Omega})$$

Radenachons

proof of thm 1 For any $(\varphi, \psi) \in \mathcal{K}$, $T \in S_{fg}$, $J(\varphi, \psi) = \int_{\Omega} -\varphi f + \int_{\Omega^*} -\psi g$ $\left(\int_{\Omega} \varphi h, T = \int_{\Omega^*} \psi h \right)$
 $\Rightarrow \sup_{\mathcal{K}} J(\varphi, \psi) = \inf_{S_{fg}} \mathcal{C} = \int_{\Omega} -\varphi f + \int_{\Omega} (-\psi \circ T) f$
 $= \int_{\Omega} -[\varphi_{T^*} \psi(Tx)] f$

(ii) $\Rightarrow$ (iii) $(u, v) \in \mathcal{K}$, $T_u \in S_{fg}$
 $\downarrow$
 $J(u, v) = \int -(u(x) + v(T_u x)) f$
 $= \int C(x, T_u x) f \quad \text{by } \chi_{u(x)}$
 $= \mathcal{C}(T_u)$

$\left[J(u, v) \leq \sup_{\mathcal{K}} J \leq \inf_{S_{fg}} \mathcal{C} = \mathcal{C}(T_u) = J(u, v) \right] \leq \sup_{\mathcal{K}} J$
 $\quad \quad \quad \text{全取号}$

(iii) $\Rightarrow$ (i) $\checkmark$

(i) $\Rightarrow$ (ii) Fix $h \in C(\Omega^*)$ and $\varepsilon \in (0, 1)$

Consider $\mathcal{V}_{\varepsilon}(y) = \mathcal{V}(y) + \varepsilon h(y)$

$u_{\varepsilon}(y) = \sup_{g \in \Omega^*} \{ -C(x, y) - \mathcal{V}_{\varepsilon}(y) \} = \mathcal{V}_{\varepsilon}^C$

We want to $u_{\varepsilon}(x) = u(x_0) - \varepsilon h(Tu(x_0)) + o(\varepsilon)$ if u is differentiable at x_0 .

On one side, take $y_0 = \chi_u(x_0) = Tu(x_0)$

$0 = u(x_0) + \mathcal{V}(y_0) + C(x_0, y_0)$
 $= u(x_0) + \underbrace{\mathcal{V}_{\varepsilon}(y_0)} - \varepsilon h(y_0) + \underbrace{C(x_0, y_0)}$
 $= \underbrace{u(x_0) - u_{\varepsilon}(x_0)} - \varepsilon h(y_0)$

On the other hand take $y_{\varepsilon} \in \chi_{u_{\varepsilon}}(x_0) \rightsquigarrow u_{\varepsilon}(x_0) + \mathcal{V}_{\varepsilon}(y_{\varepsilon}) = -C(x_0, y_{\varepsilon})$

$0 = u(x_0) + \mathcal{V}(y_{\varepsilon}) + C(x_0, y_{\varepsilon})$
 $= u(x_0) + \mathcal{V}_{\varepsilon}(y_{\varepsilon}) - \varepsilon h(y_{\varepsilon}) + C(x_0, y_{\varepsilon})$
 $= \underbrace{u(x_0) + [-u_{\varepsilon}(x_0)]} - \varepsilon h(y_{\varepsilon})$

$\downarrow$ x_0 处可微
 $\underline{y_{\varepsilon} \rightarrow y_0}$

$$v_2^c \xrightarrow{v + zh} v(x_0) - v_2(x_0) - z h(y_0) + z (\underbrace{h(y_0) - h(y_2)}_{=0(1)})$$

$$0 = \frac{d}{ds} \bigg|_{s=0} J(u_2, v_2)$$

$$= \frac{d}{ds} \int -u_2 f + \int -v_2 g$$

$$= \int_{\Omega} h(Tu) f - \int_{\Omega^*} h g$$

□

proof existence of OT. $\leftarrow T_u(x) = |D_x C(x, \cdot)|^T (-Du(x)) / \chi_u = \{y \mid u(x) + v(y) = C(x, y)\}$

It suffices to show $\exists (u, v)$, $u(x) = v^c(x) = \sup_{y \in \Omega^*} \{-C(x, y) - v(y)\} \leftarrow C$ -transform

$$s.t. J(u, v) = \sup_{(\varphi, \psi) \in K} J(\varphi, \psi)$$

$$\text{where } J(\varphi, \psi) = \int_{\Omega} (-\varphi) f + \int_{\Omega^*} (-\psi) g \quad K = \{(\varphi, \psi) \in \mathcal{C}(\bar{\Omega}) \times \mathcal{C}(\bar{\Omega}^*) : -\varphi(x) - \psi(y) \leq C(x, y) \forall x \in \Omega, y \in \Omega^*\}$$

Let (u_k, v_k) be a maximizing sequence

$$J(u_k, v_k) \longrightarrow \sup_{(\varphi, \psi) \in K} J(\varphi, \psi)$$

$$\text{Let } (\varphi, \psi) \in K \quad -C(x, y) - \psi(y) \leq \varphi(x) \quad \rightsquigarrow \sup_{y \in \Omega^*} \{-C(x, y) - \psi(y)\} = \varphi^c(x) \leq \varphi(x)$$

$$\downarrow$$

$$(\varphi^c, \psi) \quad J(\varphi^c, \psi) = \int -f \varphi^c + \int -g \psi$$

$$\geq \int -f \varphi + \int -g \psi = J(\varphi, \psi)$$

$$\downarrow$$

$$(\varphi^c, \psi^{cc})$$

$$\sup_{x \in \Omega} \{-C(x, y) - \psi^c(x)\}$$

$$J(\varphi^c, \psi^{cc}) \geq J(\varphi^c, \psi) \geq J(\varphi, \psi)$$

We find that $(u_k, v_k) \rightsquigarrow (v_k^c, v_k^{cc})$ still maximizing sequence
 $\swarrow$
 C -convex

Consider $u_k^* = v_k^c - v_k^c(x_0)$ fix. an $x_0 \in \Omega$

$$v_k^* = v_k^{cc} + v_k^c(x_0) \quad \leftarrow (v_k^* = (u_k^*)^c)$$

(u_k^*, v_k^*) is maximizing sequence.

$$J(u_k^*, v_k^*) = \int_{\Omega} (-v_k^c + v_k^c(x)) f + \int_{\Omega^*} (-v_k^{cc} - v_k^c(x)) g$$

$$\int_{\Omega} f = \int_{\Omega^*} g \implies \int_{\Omega} -v_k^c f + \int_{\Omega^*} -v_k^{cc} g = J(v_k^c, v_k^{cc})$$

We have $\|u_k^*\|_{Lip(\Omega)} \leq C$, $\|u_k^*\|_{L^\infty(\Omega)} \leq C$

$\{ \leq \sup_{y \in \Omega^*} \|D_x C(x, y)\|_{L^\infty(\Omega)} \|u_k^*\|_{Lip(\Omega)} \leq \sup_{y \in \Omega^*} \|D_x C(x, y)\|_{L^\infty(\Omega)} \|u_k^*\|_{Lip(\Omega)} \frac{\text{diam } \Omega}{\text{bounded}} \}$

$u_k^*(x) \leq u_k^*(x_0) + \dots$

$$\|v_k^*\|_{Lip(\Omega^*)} \leq C, \quad \|v_k^*\|_{L^\infty(\Omega)} \leq C$$

$v_k^* = (u_k^*)^c$

AA Thm

$$(u_k^*, v_k^*) \rightarrow (u, v), \quad J(u, v) = \sup_K J(u_k^*, v_k^*)$$

$(J(v^c, v) \geq J(u, v))$

$\hat{u}$

<u>Rmk:</u>	(u, v) maximizer	concl $\nearrow$	$\exists C, \tilde{u} = u + C$
	$(\tilde{u}, \tilde{v})$ maximizer		$\tilde{v} = v + C$

Isoperimetric inequality (Argument due to Gromov Trudinger)

$(\Omega, \frac{1_{\Omega}}{|\Omega|})$ $\xrightarrow{T = Du}$ $(B_1, \frac{1_B}{|B|})$

Brenier's map (optimal transport)

$C(x, y) = -x \cdot y$

$C(x, y) = -x \cdot y$

$Tu = Du(x)$

$\Phi_x C(x, y) = -y$

C -convex

(a) $|Du(x)| \leq 1$

(b) $\det D^2 u = \frac{|B_1|}{|\Omega|}$

$T \in S_{f,g}$, $\forall \omega^* \subseteq \Omega^*$ Borel set

$$\int_{T^{-1}(\omega^*)} f = \int_{\omega^*} f$$

$$\int_{\omega} f = \int_{\omega^*} g = \int_{Du(\omega)} g = \int_{\omega} \det D^2 u \, g(\nabla u)$$

area formula

$$f = \det D^2 u \, g(\nabla u)$$

$$\frac{|B_1|^{\frac{1}{n}}}{|\Omega|^{\frac{1}{n}}} |\Omega| = \int_{\Omega} (\det D^2 u)^{\frac{1}{n}} \leq \int_{\Omega} \frac{1}{n} \Delta u = \frac{1}{n} \int_{\partial \Omega} \langle Du, \nu \rangle \leq \frac{1}{n} |\partial \Omega|$$

✓

Rigidity. If Ω attains isoperimetric inequality $\implies \Omega = B_r(x_0)$

(i) Ω is connected ($\Omega = \Omega_1 \cup \Omega_2$, $|B_1|^{\frac{1}{n}} (|\Omega_1| + |\Omega_2|)^{1-\frac{1}{n}} = \frac{1}{n} (|\partial \Omega_1| + |\partial \Omega_2|)$)

$$(a+b)^{1-\frac{1}{n}} \geq a^{1-\frac{1}{n}} + b^{1-\frac{1}{n}} \quad \text{"a-o on b-o"} \quad \geq |B_1|^{\frac{1}{n}} (|\Omega_1|^{1-\frac{1}{n}} + |\Omega_2|^{1-\frac{1}{n}})$$

$$D^2 u \sim (\lambda_1, \dots, \lambda_n) \quad \lambda_1 = \dots = \lambda_n$$

$$D^2 u = \frac{\Delta u}{n} I = \frac{|B_1|^{\frac{1}{n}}}{|\Omega|^{\frac{1}{n}}} I$$

$$\implies D(Du - \frac{|B_1|^{\frac{1}{n}}}{|\Omega|^{\frac{1}{n}}} x) = 0 \implies T = Du(x) = \frac{|B_1|^{\frac{1}{n}}}{|\Omega|^{\frac{1}{n}}} x - x_0$$

$$\Omega = T^{-1}(B_1) = B_r(x_0)$$

Ω is connected

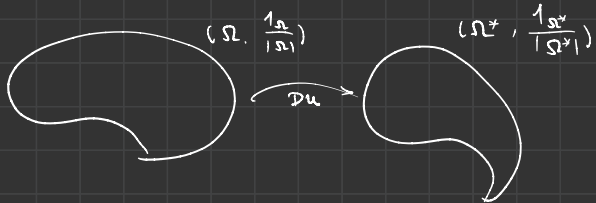
$$|\Omega + \Omega^*|^{\frac{1}{n}} \geq |\Omega|^{\frac{1}{n}} + |\Omega^*|^{\frac{1}{n}} \quad \Omega + \Omega^* = \{x+y : x \in \Omega, y \in \Omega^*\}$$

Remark: Take $\Omega^* = 2B_1$, $|\Omega + 2B_1|^{\frac{1}{n}} \geq |\Omega|^{\frac{1}{n}} + 2|B_1|^{\frac{1}{n}}$

$$|\Omega_2|^{\frac{1}{n}} = |\Omega|^{\frac{1}{n}} + \square 2 + o(2)$$

$$\frac{1}{n} |\Omega|^{\frac{1}{n}-1} \left(\frac{d}{dz} |\Omega_2| \right) \geq |B_1|^{\frac{1}{n}}$$

✓



$$|\Omega + \Omega^*| \geq \int_{\Omega} dy \geq \int_{\Omega} \det(1 + D^2 u) dx \geq \int_{\Omega} \left[1 + \det^{\frac{1}{n}} D^2 u \right]^n = \left(1 + \frac{|\Omega^*|^{\frac{1}{n}}}{|\Omega|^{\frac{1}{n}}} \right) |\Omega|^{\frac{1}{n}}$$

$y = Du(x)$ *change of var.*

$$\det(A+B) \geq \det^{\frac{1}{n}} A + \det^{\frac{1}{n}} B$$

($A, B \geq 0$) (Exercise)

□

3. Geometric optics

$$C(x, y) = \begin{cases} -\log x \cdot y & x, y \in S^n \leftarrow \text{Alexandrov (1940s)} \\ \log(1 - x \cdot y) & x, y \in S^n \leftarrow \text{Reflector design} \\ \log(1 - \kappa x \cdot y) & \kappa = 1 \leftarrow (x, -J \cdot \text{Wang}) \\ -\log(\kappa x \cdot y - 1) & \kappa > 1 \leftarrow \text{Refraction design} \end{cases}$$

(Guillemin - Maug 2007)

Villani, Otlet and New. OT.

Gluedo - Figollin An invitation to OT wasserstein distance and gradient flows.

$$T_u(x) = \left[D_x C(x, \cdot) \right]^{-1} (-Du(x)) \iff D_x C(x, T_u(x)) = -Du(x)$$

$$C_{ij}(x, T_u(x)) + C_{\alpha j} T_i^\alpha = -u_j$$

$$\Rightarrow D^2 u + D_x^2 C = -D_x^2 C D T$$

$$\Rightarrow \det(D^2 u + D_x^2 C(x, T_u(x))) = \left| \det D_x^2 C(x, T_u(x)) \right| \frac{f(x)}{g(T_u(x))}$$

Monge - Ampère

$$C_x(x, T_u(x)) = -u_i$$

one-to-one $T_{\#} f = g$

$D^2u(x) + D_x^2 C(x, y) \geq 0$ attains minimal at x

$$\partial \in \chi_u(x) \quad \uparrow \quad T_u(x)$$

• $C(x, y) = -x \cdot y \quad \det D^2u = \frac{f(x)}{g(Du(x))} \quad Du(x) = \Omega^*$

(i) Delaunay ($n=2$) Urbas ($n \geq 2$). uniformly convex domains: $f, g > 0 \in C^{1,1}$
 $u \in C^{2,\alpha}(\bar{\Omega})$

(ii) C. Cabré uniformly convex $f, g \in C^\alpha(\bar{\Omega}) \Rightarrow u \in C^{2,\alpha}(\Omega)$

(iii) C. Chen, J. K. Lin - X. J. Wang convex domains $f, g \in C^\alpha(\bar{\Omega}) \Rightarrow u \in C^{2,\alpha}(\Omega)$

$n=2$

$$\det D^2u = \begin{vmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{vmatrix} = u_{11}u_{22} - u_{12}u_{21} = \underbrace{a_{11}u_{11}}_{\frac{1}{2}u_{22}} + \underbrace{a_{22}u_{22}}_{\frac{1}{2}u_{11}} + \underbrace{a_{12}u_{12}}_{-\frac{1}{2}u_{12}} + \underbrace{a_{21}u_{21}}_{-\frac{1}{2}u_{21}} = \sum_{1 \leq i, j \leq 2} a_{ij}u_{ij}$$

$$a_{ij} = \begin{pmatrix} \frac{1}{2}u_{22} & -\frac{1}{2}u_{12} \\ -\frac{1}{2}u_{21} & \frac{1}{2}u_{11} \end{pmatrix}$$

$|D^2u| = C$ uniformly elliptic

• General $C(x, y) \quad \det(D^2u - A(x, Du)) = \varphi(x, Du(x))$
 $- \tilde{A}_{\alpha\beta}(x, Tu(x))$

X. N. Ma, Trudinger - X. J. Wang $D_{P_i P_j}^2 A_{ikl} \xi_i \xi_j \eta_k \eta_l \geq C_0 |\xi|^2 |\eta|^2 \quad \xi \perp \eta$
 $(C_0 > 0)$
 $C = \frac{1}{2} d^2 C(x, y)$

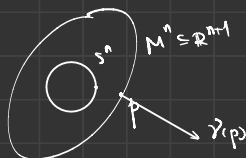
larger necessary

Figalli, Kim, McCormick, J. K. Lin, Trudinger, X. J. Wang, ...

$C = |x - y| \quad \Omega, \Omega^* \subseteq \mathbb{R}^2 \quad T \text{ is continuous}, \quad T \notin C^{\frac{2}{3}+2} \quad \forall \varepsilon > 0.$

(Colombo, Ishii $T \notin C^{\frac{1}{2}+2}$) $T \in \text{Lip}(\Omega \setminus \text{null set})$ n=2

Minkowski problem (prescribing Gauss curvature of convex hypersurface)



$$\gamma: M^n \rightarrow S^n$$

$$p \mapsto \gamma(p)$$

Given \$f \in C^\infty(S^n)\$ Gauss curvature.

$$\exists M \subseteq \mathbb{R}^{n+1} \text{ s.t. } K(\gamma^{-1}(x)) = f(x)$$

- Minkowski (1900s) discrete measure version
- Fenchel - Jensen Aleksandrov variational problem
- Nirenberg Pogorelov (1975) \$n=2\$
- Chang - Yau (1970s) Pogorelov \$n \geq 2\$

Minkowski problem Nirenberg Pogorelov Chang - Yau

Thm If \$f \in C^2(S^n)\$, \$f > 0\$ and \$\int_{S^n} \frac{x^i}{f} dx = 0\$ \$x = (x^1, x^2, \dots, x^n)\$
 then \$\exists\$ uniformly ^{closed} convex hypersurface \$M^n \subseteq \mathbb{R}^{n+1} \subset \mathbb{R}^{3,\alpha}\$, \$\alpha \in (0,1)\$
 s.t. \$K(\gamma^{-1}(x)) = f(x) \quad \forall x \in S^n\$

Remark: \$K = \frac{n}{\det g_{ij}} K_i\$ \$K\$ is the eigenvalue of \$h_{ij}\$ w.r.t. \$g_{ij}\$, \$\det(h_{ij} - K g_{ij}) = 0\$

$$= \frac{\det h_{ij}}{\det g_{ij}}$$

This condition is necessary

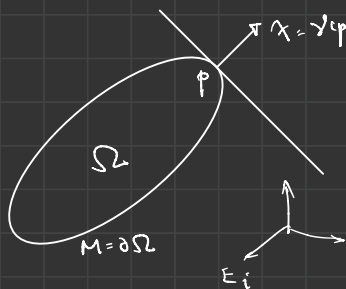
$$\int_{S^n} \frac{x^i}{f} dx = \int_{S^n} \frac{x^i}{K} \boxed{\frac{dx}{dp}} dp = \int_M \gamma \cdot E_i d\mu$$

$$= \int_\Omega d\psi E_i$$

$$= 0$$

Gauss map's Jacobian

$$K = \text{Jac}$$



Define \$U: S^n \to \mathbb{R}\$, \$x \mapsto U(x) = \frac{\langle \gamma^{-1}(x), x \rangle}{X(x)} = \sup_{z \in \Omega} z \cdot x\$
 带 \$\Gamma\$ 相球面上能量

Lemma: If $X: S^n \rightarrow M \subseteq \mathbb{R}^{n+1}$ $X(x) = \nabla u(x) + u(x) \cdot \lambda$
 $x \mapsto v^{-1}(x)$ 法向 法向

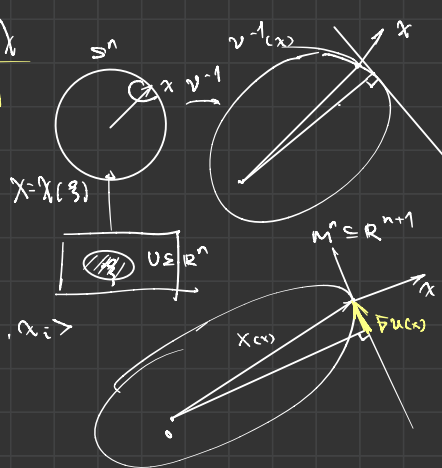
• $u_i = \langle X_i, x \rangle = \langle X, x_i \rangle$

• Assume $X = A^k x_k + u(x) \lambda$
 $\uparrow$
 $x^\perp$

$$\langle x, x_i \rangle = A^k x_k \cdot x_i \Rightarrow A^k = g^{-ki} \langle x, x_i \rangle = g^{-ki} u_i$$

$$X = \underbrace{g^{-kl} u_i x_k}_{= \nabla u} + u \lambda$$

□



Lemma $\frac{1}{K(v^{-1})} = \frac{\det(\nabla^2 u + u \bar{g})}{\det \bar{g}}(x) \Rightarrow \frac{\det(\nabla^2 u + u \tilde{g})}{\det \tilde{g}} = \frac{1}{f}$

Proof $u_{ij} = \langle X, \frac{x_i x_j}{\bar{g}_{ij}} \rangle + \langle x_j, x_i \rangle$ $x_i = x_{g_i}$ (坐标分量)

m : Weingarten formula

S^n Gauss formula

$$= \Gamma_{ij}^k u_k - \bar{g}_{ij} u + h_i^k g_{kj} - h_{ij}$$

unbiloc
全张

$\Rightarrow \boxed{\nabla_{ij}^2 u + u \bar{g}_{ij} = h_{ij}}$

$$\bar{g}_{ij} = \langle x_i, x_j \rangle = \langle h_i^k x_k, h_j^l x_l \rangle = h_i^k g_{kl} h_j^l = h_i^k h_j^l g_{kl} = h_i^k h_j^l \underbrace{g_{kl}}_{\text{metric of } (g_{kl})}$$

$$\leadsto \frac{\det(\nabla u + u \bar{g})}{\det \bar{g}} = \frac{\det h}{(\det h)^2 (\det g)^{-1}} = \frac{\det g}{\det h} = \frac{1}{K(v^{-1})} \quad \square$$

Corollary Main theorem $\Leftrightarrow$ Find $u \in C^{3,\alpha}(S^n)$, $\nabla^2 u + u \bar{g} > 0$

s.t. $\frac{\det(\nabla^2 u + u \bar{g})}{\det \bar{g}} = \frac{1}{f} \quad (*)$

$$x = (x^1, \dots, x^{n+1}) \in \bigcup_{i=1}^n S_i$$
$$S_m = \left\{ \omega \in H^m(S^n), \int \omega x^i dx = 0 \right\}$$

integral by pole

$$\frac{1}{n} \int (x_j^k + x_{s,j}^k) \psi_j^s w$$

$\therefore$ it's well-defined.

$$\int \lambda^k \det(\underbrace{\nabla^2 w + \omega I}_{W''}) \neq 0$$

Define $I = \{t \in [0, 1], \varphi_t(u) = 1 - t + t \frac{1}{f}\}$ admits $C^{3,\alpha}$ uniformly convex solution φ

(ii) I is open. $\exists t_0 \in I$. $u = u^{t_0}$. $\mathcal{U}(u) = 1 \cdot t_0 + t_0/f = f_{t_0}$. $\exists t$ is closed to t_0 .

$$\mathcal{L}_{u^0}(\phi) = \frac{d}{dx} \Big|_{x=0} 4(u^0 + 2\phi) - 4u^0 (\bar{v}_j^0 \phi + \phi \bar{v}_{-j}^0)$$

利州元年子。

1

可验证自伴性

$$\exists t_j \text{ s.t. } \det(\nabla^2 u^{t_j} + u^{t_j} I) = f_{t_j}$$

$$t_j \rightarrow t_* \iff \|u^{t_j}\|_{C^{2,\alpha}(S^n)} \leq C \quad (\text{independent of } j)$$

$$u^y \rightarrow u_a$$

C^0 -estimate $M \rightarrow M + \vec{a} \quad u \mapsto u_a = u(x) + aX$

$$\exists \underbrace{\vec{a} \in \mathbb{R}^{n+1}}_{(a_1, \dots, a_{n+1})} \text{ s.t. } \int_{S^n} u_a(x) dx = 0 \quad \left(0 = \int_{S^n} u x^i dx + \int_{S^n} \underbrace{(a \cdot x) x^i}_{a_1 x^1 + \dots + a_{n+1} x^{n+1}} \right)$$

We always assume $\int_{S^n} u_0 x^i dx = 0 \quad \forall i$

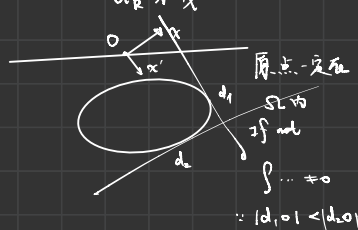
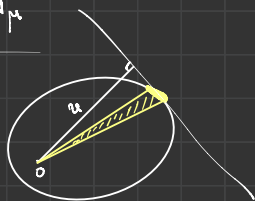
$$|S_1| = \frac{1}{n+1} \int_{S^n} \frac{u}{K} = \int_{S^n} \frac{u}{f} \frac{1}{n+1}, \quad \frac{1}{K} dx = d\mu$$

$$\geq c \int_{S^n} u$$

$$= C \int u$$

$$\{x \cdot x_0 < 0\} \cup \{x \cdot x_0 \geq 0\}$$

Assume $u(x_0)$
 $= \max_{S^n} u$



$$\int_{S^n} \frac{1}{K} dK = \int_{S^n} \frac{1}{f}$$

$$\int_M d\mu = |M|$$

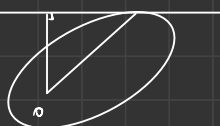
$$\geq C \int u$$

$$\{x \cdot x_0 > 0\}$$

$$\geq C u(x_0) \int (x \cdot x_0)_+$$

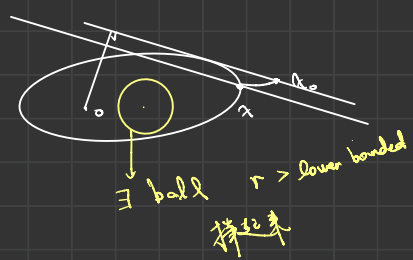
$$\geq C' u(x_0)$$

$$u(x) \geq u(x_0) x_i x_j$$

$$u(x) = \sup_z x_i z$$


$$\frac{1}{2} d \leq u(x_0) \leq C |\Omega| \leq C |\partial \Omega|^{\frac{n}{n-1}} \leq C_n f$$

(等号不成立)



$$0 \leq \min \leq \max \leq C$$

$$d \leq C |\Omega| \leq C \int d^n \Rightarrow f_- \geq \frac{C}{d^{n-1}} \geq C'$$

δ (strictly lower bounded)

e^1 - estimates

$$X = \nabla u + u X \Rightarrow |\nabla u| \leq \frac{|X|}{d} + \frac{|u X|}{\|u\|_{L^\infty}} \leq C$$

e^2 - estimates (using maximum principle)

$$\|\nabla^2 u\| \leq C \Leftrightarrow \max_{S^n} \lambda_{\max} \leq C$$

$\lambda_{\max}$ is the largest eigen values of $\nabla^2 u + u I$

$(\lambda_1, \dots, \lambda_n)$

$\min \lambda_1 \dots \lambda_n = \frac{1}{f} \leq 1$

$$\nabla^2 u + u I \geq \delta_0 I$$

Consider

$$G = \sum \lambda_i$$

$$= \Delta_{S^n} u + n u$$

$$u_{k\ell, i} = u_{ki, \ell} + \bar{R}_{k\ell i}^h u_h$$

Ricci identity

$$G_i = (u_{k\ell} + n u)_i = u_{k\ell i} + n u_i$$

$$G_{ij} = u_{k\ell, ij} + n u_{ij}$$

$$= u_{ki, \ell j} + \bar{R}_{k\ell i}^h u_{hj} + n u_{ij}$$

$$= u_{kij} + \dots$$

$$= u_{ij k} + \dots$$

$$\log \det (u_{ij} + u \delta_{ij}) = \int$$

$$\widehat{b}^{ij} (u_{ij, k} + u_{k, ij}) = \widehat{f}_k$$

$$(\log \det b_{ij})' = \frac{\partial \log \det b_{ij}}{\partial b_{ij}}$$

$\widehat{b}^{ij}$

$$G_{ij} = \Delta_S^n b_{ij} + G g_{ij} - n b_{ij}$$

$$0 \geq \tilde{b}^{ij} G_{ij} = \tilde{b}^{ij} \Delta_S b_{ij} + G \tilde{b}^{ij} g_{ij} - n^2$$

$$\geq -C_0 + G C_1 \xrightarrow{G \rightarrow \infty}$$

$$G(x) = \max G(x) \leq C n f$$

再求导数

$$\tilde{b}^{ij} (u_{ij,kk} + u_{kk} \delta_{ij})$$

$$\tilde{b}^{ij} b_{j\ell} = \delta_\ell^i$$

$$\partial \tilde{b}^{ij} b + \tilde{b}^{ij} \partial b = 0$$

$$\therefore \partial \tilde{b}^{ij} = -\tilde{b}^{ik} \partial b^{jl}$$

$$- \left(\tilde{b}^{ip} \tilde{b}^{jq} \frac{\partial b_{pq}}{\partial b_{rs}} b_{rs,k} \right)$$

$$\tilde{b}^{ip} \tilde{b}^{jq} b_{rs,k} b_{ij,k} = \tilde{f}_{tk}$$

$\Delta_S^n b_{ij}$

$$\leq C$$

$$\|u\|_{C^1(S^n)} \xrightarrow{\quad} \|u\|_{C^{2,\alpha}(S^n)} \leq C$$

Evans - Krylov

$$\log - \det(\nabla^2 u + u I) = -\log f = f$$

$$\tilde{b}^{ij} (\nabla_{ij} u_e + u_e \delta_{ij}) = \hat{f}_e \Rightarrow u_e \in C^{2,\alpha}$$

"closed" $\square$