Letue 1. $\mathcal{L}u = \sum \underset{\text{对称的}}{\underline{a_{ij}}} u_{ij} + \sum b_i u_i + cu = f$ in $\Omega \subseteq \mathbb{R}^n$

Elliptic. $(a_{ij})(x) > 0$ 正定 $\forall x \in \Omega$

<u>uniformly elliptic</u> $\underset{\sim}{\lambda I} \le (a_{ij}(x)) \le \underset{\sim}{\Lambda I}$ $0 < \lambda \le \Lambda$ $\forall x \in \Omega$

<u>Examples</u>. $\Delta u = 0$

"足够简单、足够复杂" → 运用更多 case.

Consider $a_{ij} u_{ij} = f$ $0 < \lambda I \le (a_{ij}(x)) \le \Lambda I$ $\forall x \in \Omega$

√

- Schauder estimates (1930s) C^α: $\sup\limits_{x \ne y} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^\alpha} \le C$

  $a_{ij} \in C^\alpha(\Omega), f \in C^\alpha(\Omega) \Rightarrow u \in C^{2\alpha}(\Omega)$

  系数足够好，方程正则性就好！

- Calderon - Zygmund estimates. (1950s)

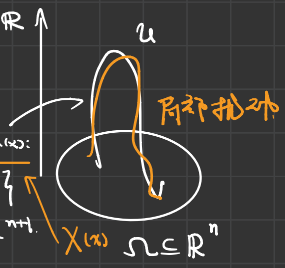
  $a_{ij} \in C^0(\Omega)$, $f \in L^p(\Omega)$ $(p > 1)$. $\rightsquigarrow u \in W^{2,p}(\Omega)$.

Assume 0边值.

$$\text{Area}(M_u) = \int_\Omega \sqrt{\det g_{ij}}$$

$$= \int_\Omega \sqrt{1 + |\nabla u|^2}$$

$M_u = \{\underline{(x, u(x))}: x \in \Omega\} \subseteq \mathbb{R}^{n+1}$.



$$g_{ij} = \delta_{ij} + u_i u_j$$

<u>Define</u>: $F(\xi) = \sqrt{1 + |\xi|^2}$

$$\text{Area}(M_u) = \int_\Omega F(\nabla u)$$

$$g_{ij} = \langle X_i, X_j \rangle$$

$$= \langle e_i + u_i e_{i+1}, e_j + u_j e_{j+1} \rangle$$

"极小曲面"

$$0=\frac{d}{dt}\Big|_{t=0}\text{Area}(M_{u+t\phi})=\frac{d}{dt}\Big|_{t=0}\int_\Omega F(\nabla u+tD\varphi)$$

$\forall\ \varphi\in C_c^\infty(\Omega)$ $$=\int_\Omega F_{\xi_i}(\nabla u)\,\varphi_i=-\int_\Omega\left[\partial_i F_{\xi_i}(\nabla u)\right]\varphi$$

$$\Rightarrow\ \boxed{\partial_i\big(F_{\xi_i}(\nabla u)\big)=0.}$$ ，正则性如何？

$$\Updownarrow$$

$$\boxed{\underbrace{F_{\xi_i\xi_j}(\nabla u)}_{a_{ij}(x)}u_{ij}=0}\iff F_{\xi_i\xi_j}=\frac{1}{\sqrt{1+|\xi|^2}}\left(\delta_{ij}-\frac{\xi_i\xi_j}{1+|\xi|^2}\right)$$

$u\in C^{0,1}(\Omega)$ 时 ↗ Bounded. uniformly elliptic.

$\downarrow$ We hope $u\in C^{1,\alpha}$ to apply Schauder estimates

Suppose $u\in C^{1,\alpha}\xrightarrow[\text{Schauder estimate}]{F_{\xi_i\xi_j}u_{ij}=0}u\in C^{2,\alpha}$

两边求导，$$\underset{\substack{\cap\\ C^\alpha}}{\underline{F_{\xi_i\xi_j}(\nabla u)}}(\boxed{u_e})_{ij}+\underset{\substack{\cap\\ C^\alpha}}{\underline{F_{\xi_i\xi_j\xi_k}(\nabla u)}}\,\underset{\substack{\cap\\ C^\alpha}}{\underline{u_{ij}}}(\boxed{u_e})_k=0$$ "e方向" u_e

∵F 对 ξ 光滑.

$$\xrightarrow[\text{estimate}]{\text{Schauder}}u_e\in C^{2,\alpha}\rightsquigarrow u\in C^{3,\alpha}\xrightarrow[\text{bootstrap}\cdots\ \text{argument.}]{}u\in C^{k,\alpha}\ \forall k$$

Fix $e\in S^{n-1}$. $\partial_i(F_{\xi_i\xi_j}(\nabla u)\,\partial_j\boxed{u_e})=0$

✓ • De Giorgi (1957?) – Moser (1960?) | Nash (parabolic 1958?)

"divergence" $\partial_i(a_{ij}(x)\,\partial_j u)=0$. $a_{ij}\in L^\infty\Longrightarrow u\in C^\alpha(\Omega)$

(?) Krylov – Safanov (1980?)

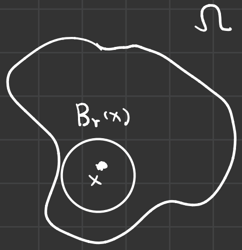
$$a_{ij}(x)\,u_{ij}(x)=0,\quad a_{ij}\in L^\infty(\Omega)\Rightarrow u\in C^\alpha(\Omega)$$

• $F(D^2u)=0$ (eg. $F(M)=\det M$.)

Evans – Krylov F is an convex / concave. $u\in C^{1,1}\Rightarrow u\in C^{2,\alpha}$

$$\underset{\overset{\|}{a_{ij}(x)}}{F_{ij}(D^2u)}(u_e)_{ij} = 0. \xrightarrow[\text{if } u \in C^{2,\alpha}]{\text{Schauder estimates}} u \in C^{3,\alpha}$$

$\Delta u = 0$ in $\Omega \subseteq R^n$



Mean Value Property

$$u(x) = \fint_{\partial B_r(x)} u = \fint_{B_r(x)} u$$

proof: Define $\phi(r) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y)\,dy$

If $\phi'(t) \equiv 0$, for $t \in (0, r)$,

then we are done.

Exercise (usig integration by parts)

$y = rz + x$. $z \in \partial B_1$.

$$\int_{B_r(x)} u(y)\,dy = \int_0^r \underbrace{\boxed{\int_{\partial B_t(x)} u(z)\,dz}}_{u(x)|\partial B_t|} dt = u(x)\int_0^r |\partial B_t|\,dt = u(x)|B_r(x)|$$

□

Rmk: (i) 上性质为 "$\Delta u = 0$" 等价刻画. $u \in C^2$

(ii) $\Delta u \geq 0$. $u(x) \leq \fint_{\partial B_r(x)} u \left(\leq \fint_{B_r(x)} u\right)$ → sub-harmonic

$\Delta u \leq 0$. $u(x) \geq \cdots \ (\geq \cdots)$ → super-harmonic

$\boxed{\text{Ex}}$: $\int_M u\, d\text{Vol}$

↓ Riemann 体积元. $d\text{Vol} = \sqrt{g_{ij}}$

↓

Remark: ① Let (M,g) be a complete Riemann mnfd. $\underline{Ric \geq 0}$

Then $u \geq 0$. $+ \Delta u \leq 0$. $\Rightarrow$ $u(x) \geq \frac{1}{|\hat{B}_r|} \int_{B_r(x)} u \, dvol_g$, $r \in (0, \underset{\uparrow \text{单射半径}}{inj(x)})$

欧氏球 ↙ ↘ 测地球

$\phi' = ?$ 用体积比较定理.

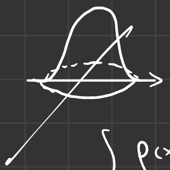
② . - - - - - $\overset{\text{截曲率}}{Sect} \leq K_0$

$u \geq 0 + \Delta u \geq 0 \Rightarrow u(x) \leq \frac{1}{|B_r^{K_0}|} \int_{B_r(x)} u \, dvol_g$

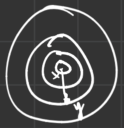
Interior estimates

Consider $\rho \geq 0$. $\in C_c^\infty(\mathbb{R}^n)$ $\text{Supp}\, \rho = B_1$. $\int_{\mathbb{R}^n} \rho = 1$

$\rho(x) = \eta(|x|)$ Examples. $\eta(t) = \begin{cases} C e^{-\frac{1}{1-t^2}} & t \in [0,1) \\ 0 & t \geq 1 \end{cases}$



$$\int_{\mathbb{R}^n} \rho(x-y) u(y) \, dy = \int_0^{+\infty} \int_{\partial B_t(x)} \eta(t) u(z) \, dz \, dt$$



suppose u has the "MVP"

$$= \int_0^\infty \eta(t) \, u(x) \, |\partial B_t(x)| \, dt$$

$$= u(x) \int_0^\infty \underbrace{\eta(t)}_{\rho} |\partial B_t(x)| \, dt$$

$$= u(x) \int_{\mathbb{R}^n} \rho = u(x).$$

Thm If u satisfies the mean value property in Ω

then $\forall B_r(x) \subset \Omega$ we have $|D^k u(x)| \leq \frac{C_{n,k}}{r^{n+k}} \|u\|_{L^1(B_r(x))}$

Proof: Take such ρ. Then. $u(x) = \int_{\mathbb{R}^n} \rho(x-y) u(y) \, dy$.

$$\Rightarrow |D^k u(x)| = \left| \int_{\mathbb{R}^n} D_x^k \rho \, u(y) \right| \leq \underset{\substack{\downarrow \\ C_{n,k}}}{\underline{C_{n,k,\rho}}} \|u\|_{L^1(B_1(x))}$$

Consider $\tilde{u}(y) = u(\underbrace{ry+x}_{B_r(x)})$ $y \in B_1(0)$ Check. $\tilde{u}(y)$ satisfies MVP

$$|D_y^k \tilde{u}(0)| \le C_{n,k}\,\|\tilde{u}\|_{L^1(B_1(0))} = \frac{C_{n,k}}{r^n}\|\hat{u}\|_{L^1(B_r(x))}$$

(If $\Delta u = 0$.) $\quad \shortparallel$

$$\underline{r^k}\, |D_x^k u(x)|$$

$\square$

Corollary (Liouville) If $u \in C^\infty(\mathbb{R}^n)$, $\Delta u = 0$ and u is one-sided bounded, then $u =$ const.

<u>proof</u> $\boxed{u \ge 0}$ By interior estimates. $\forall x_0 \in \mathbb{R}^n$

$$|Du| \le \frac{C_n}{r^{n+1}}\|u\|_{L^1(B_r)} \qquad \forall r > 0$$

$$\text{MVP} \rightarrow \; = \frac{C}{r}u(x) \qquad \text{letting } r \to \infty$$

$\Longrightarrow$ $Du(x_0) = 0$.

$u - \inf_{\mathbb{R}^n} u \ge 0$. $\quad \surd$

$\sup_{\mathbb{R}^n} u - u \ge 0$ $\quad \surd$

$\square$.

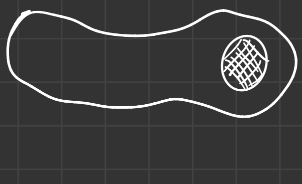
<u>Remark</u>: (M, g) be a complete non compact Rie mnfd, $\Delta u = 0$, $\mathrm{Ric} \ge 0$,

u is one-sided bounded $\Rightarrow$ $u =$ const.

<u>Proof</u>: Apply maximal principle to $\eta(x) \log|\nabla u|$

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla^2 u| + \langle \nabla\Delta u, \nabla u\rangle + \underline{\mathrm{Ric}(\nabla u)} \qquad \text{(Bochner formula)}$$

Thm (Maximum Principle) If $\Delta u \ge 0$, then $\max_{\Omega} u = \max_{\partial\Omega} u$



$u(x) \le ⨍_{B_r} u$ If $\Delta u \le 0$ then $\min_{\Omega} u = \min_{\partial\Omega}$

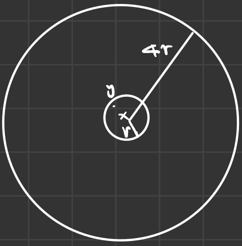
Thm (Harnack inequality) $\overset{(u>0)}{\vee}$ $\Delta u = 0$ in B_{4r}, $\forall x, y \in B_r$ $\longrightarrow \Omega' \subset\subset \Omega$

$u(x) \leq C\, u(y)$ for some a $\underset{\text{万有的 (仅与 } n \text{ 有关)}}{\underline{\text{universal}}}$ $\underline{C} > 0$

$\longrightarrow \Omega$ (general).

$\longrightarrow C(n. \Omega. \underset{\sim}{\Omega'})$

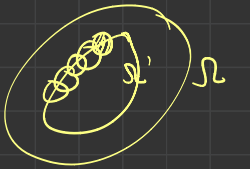
$d(\Omega, \Omega')$?



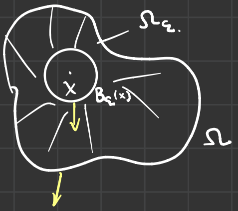
$$u(y) = \fint_{B_{3r}} u \geq C \cdot \fint_{B_r(0)} u = C\, u(0)$$

$$u(0) = \frac{1}{B_{4r}} \int_{B_{4r}} u \geq \frac{C}{B_r} \int_{B_r(x)} u = C\, u(x)$$

☐



$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$



基本解

↓

We say $G_\Omega(\cdot\,; x_0) = \Psi(\cdot\,; x_0) + H(\cdot)$ in Ω is Green function

if $$\Psi(x; x_0) = \begin{cases} -\frac{1}{2\pi} \ln|x - x_0|, & n = 2 \\ \frac{1}{(n-2)\underline{|S^{n-1}|}} \cdot \frac{1}{|x - x_0|^{n-2}}, & n \geq 3 \end{cases}$$

↘ area of S^{n-1}

and H is a hamonic function

$$\begin{cases} \Delta H = 0 & \text{in } \Omega \\ H(x) = -\Psi(x; x_0) & \text{on } \partial\Omega \end{cases}$$

$$\rightsquigarrow \begin{cases} \Delta G_\Omega(x; x_0) = 0 & \forall x \in \Omega \setminus \{x_0\} \\ G_\Omega(x; x_0) = 0 & \text{on } \partial\Omega \end{cases}$$

$$0 = \int_{\Omega_\varepsilon} u \Delta G = -\int_{\Omega_\varepsilon} \nabla u \cdot \nabla G + \int_{\partial\Omega} u \partial_\nu G - \int_{\partial B_\varepsilon} u \partial_\nu G$$

$$= \int_{\Omega_\varepsilon} \underset{f}{(\Delta u)} G - \int_{\partial\Omega} \underline{\partial_\nu u} \cdot \overset{0}{\underset{\sim}{G}} + \int_{\underset{\varepsilon^{n-1}}{\partial B_\varepsilon}} u_\nu \boxed{G} \xrightarrow{} 0 \quad \varepsilon^{2-n} \quad (n \geq 3)$$

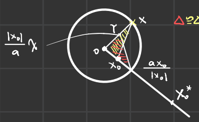
$$+ \int_{\underline{\partial\Omega}} \underset{\varphi}{u} \partial_\nu G - \boxed{\int_{\boxed{\partial B_\varepsilon}} u \underset{\varepsilon^{1-n}\ (n \geq 3)}{\underline{\partial_\nu G}}} \longrightarrow -u(x)$$

ε^{n-1}

Green . representation formula :

$$u(x_0) = -\int_\Omega f\,G - \int_{\partial\Omega} \varphi\,\partial_\nu G \qquad \forall\, x_0 \in \Omega$$

For example. B_a



怎么取？

边界上: $\frac{1}{|x-x_0|^{n-2}} - \boxed{H}$

$$\downarrow$$

$$\left|\frac{|x_0|}{a}x - \frac{x_0}{|x_0|}a\right|^{2-n}$$

$$|x-x_0|^{2-n} = \left|\frac{|x_0|}{a}x - \frac{x_0}{|x_0|}a\right|^{2-n}$$

$$= \left[\left|\frac{|x_0|}{a}\right|^{2-n}\right]\left|x - \underbrace{\frac{a^2}{|x_0|^2}x_0}_{\overset{\shortparallel}{x_0^*}}\right|^{2-n}$$

Check ← is a harmonic func ? √

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

perron method

or sub-harmonic

Define : $S_\varphi = \{$ ~~subsolution~~ $v,\ v|_{\partial\Omega} \le \varphi \}$

$$u(x) = \sup_{v \in S_\varphi} v(x)$$

↪ harmonic

Thm: Ω admits exterior ball condition

$\longrightarrow u|_{\partial\Omega} = \varphi$

↑

contructing barrier function.

Q1. well-defined ? $\inf \varphi \in S_\varphi$ $(S_\varphi \neq \phi)$, $\forall v \in S_\varphi$, $v \le \sup_{\partial\Omega} \varphi$

Fix $\bar{x} \in \Omega$. $\exists v_k \in S_\varphi$, $v_k(\bar{x}) \longrightarrow u(\bar{x})$

$$\begin{cases} \Delta h_k = 0 & \text{in } B_\delta(\bar{x}) \\ h_k = v_k & \text{on } \partial B_\delta(\bar{x}) \end{cases}$$

$$\hat{v}_k = \begin{cases} \boxed{h_k} & \text{in } B_\delta(\bar{x}) \\ v_k & \text{outside} \end{cases}$$

↓

perron 提升

H(x) (AA Thm)

$B_{\delta/2}(\bar{x})$

[A-L]

WTS $\hat{v}_k(\bar{x}) \longrightarrow u(\bar{x})$

Lect 2. Maximum principle

$u \in C^2(\Omega) \cap C(\bar{\Omega})$

Thm (weak MP) $\mathcal{L}u = a_{ij}u_{ij} + b_i u_i + cu \geqslant 0$ $(a_{ij} \geqslant \lambda\delta_{ij},\ |a_{ij}|_{C^0}.\ |b_i|_{C^0}.\ |c|_{C^0} \le \Lambda)$

Suppose $c \le 0$ in Ω Then $\sup_\Omega u \le \sup_{\partial\Omega} u^+$

$$u^+(x) = \max\{u(x), 0\}$$

proof: Assume w.l.o.g $\sup_\Omega u \geqslant 0$

Suppose $x_0 \in \Omega$. $u(x_0) = \sup_\Omega u$

$$a_{ij}\underbrace{u_{ij}(x_0)}_{\overset{\wedge\!\shortparallel}{0}} + b_i\underbrace{u_i(x_0)}_{\overset{\shortparallel}{0}} + c\underbrace{u(x_0)}_{\overset{\wedge\!\shortparallel}{0}} = \mathcal{L}u\Big|_{x_0} > 0 \qquad \text{矛盾.}$$

Consider $w = u + \varepsilon e^{\gamma x_1}$ 取 $\gamma >> 1$ (与 a_{ij}. b_i. c 相关)

$$\mathcal{L}w = \underbrace{\mathcal{L}u}_{\geqslant 0} + \varepsilon\,\underline{\mathcal{L}(e^{\gamma x_1})} > 0$$

$$= \underbrace{(a_{11}\gamma^2 + b_1\gamma + c)\,e^{\gamma x_1}}_{\geqslant 0}$$

$$\sup_\Omega u \le \sup_\Omega w \le \sup_{\partial\Omega} w$$

$$\le \sup_{\partial\Omega} u + \varepsilon \sup e^{\gamma x_1}$$

↗ 0

□

Corollary 1. $\mathcal{L}u=0$ & $c\le 0$, $\sup\limits_{\Omega}|u| \le \sup\limits_{\partial\Omega}|u|$

proof Apply MP & u. & $-u$.

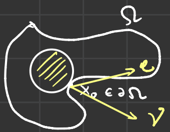
Corollary 2. Suppose $c\le 0$. if $\begin{cases}\mathcal{L}u=f\\ u=\varphi\end{cases}$ admits a $C^2(\Omega)\cap C^0(\bar\Omega)$ solution then it is unique.

proof Apply MP $v=u_1-u_2$.

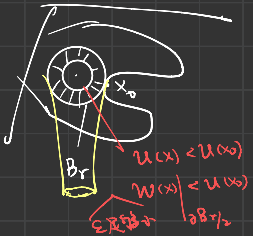
$$\begin{cases}\mathcal{L}v=0 & \text{in } \Omega\\ v=0 & \text{on } \partial\Omega\end{cases}$$

Thm (Hopf lemma) $\mathcal{L}u=a_{ij}u_{ij}+b_iu_i+cu\ge 0 \xrightarrow{\quad} C^2(\Omega)\cap C^0(\bar\Omega)$ ...

Suppose $c\le 0$ in Ω. If Ω admits an interior ball B_r at $x_0\in\partial\Omega$, and $u(x_0)>u(x)$ $\forall x\in B_r$, $u(x_0)=\sup\limits_{\Omega}u\ge 0$. then



$$\lim_{t\to 0^+}\frac{u(x_0)-u(x_0-te)}{t}>0 \quad \forall e\in S^{n-1}.\ e\cdot\nu>0.$$



Proof. 1° construct a function h in

$$T=B_r\setminus B_{r/2}. \quad \begin{cases}\mathcal{L}h\le -1 & \text{in } T\\ h\big|_{\partial B_r}=0, & h\big|_{\partial B_{r/2}}<0\end{cases}$$

$$h(x)=h(\rho),\quad \rho=|x|\in(\tfrac{r}{2},r)$$

$$h'\ge 0,\quad h''\le 0,\quad h\le 0$$

$$\mathcal{L}h=a_{ij}h_{ij}+b_ih_i+ch$$

$$=a_{ij}\left(h'\left(\frac{\delta_{ij}}{\rho}-\frac{x_ix_j}{\rho^3}\right)+h''\frac{x_ix_j}{\rho^2}\right)+b_ih'\frac{x_i}{\rho}+ch$$

$$\le \lambda h''+\underset{\frac{r}{2}\swarrow}{\frac{h'}{\rho}}\Big[\underset{\underset{n\Lambda}{\wedge\!\!\mid}}{a_{ij}}-\boxed{\frac{a_{ij}x_ix_j}{\rho^2}}+\underset{\underset{\Lambda r}{\wedge\!\!\mid}}{b_ix_i}\Big]+ch$$

$$\le \lambda h''+4n\Lambda h'-\Lambda h'=-1,\ \text{solve}\Rightarrow h=e^{\lambda\rho}$$

参数.

$$h(\rho)=\frac{1}{\Lambda}+c_1e^{\theta_1\rho}+c_2e^{-\theta_2\rho}$$

$\theta=\underset{>0}{\underline{\theta_1}}, -\underset{>0}{\underset{\sim}{\theta_2}}$ are roots of $\lambda\theta^2+4n\Lambda\theta-\Lambda=0$

舍去一个不合适的.

$$\begin{cases}h(\rho)=\frac{1}{\Lambda}-C_2e^{-\theta_2\rho}\\ h(r)=0.\end{cases}$$

$$\Rightarrow h(\rho) = \frac{1}{\lambda}\left[1 - e^{-\theta_2(r-\rho)}\right], \qquad \underset{\text{boundary}}{h'(r)} = \frac{1}{\lambda}\theta_2 > 0$$

2° Completing the proof

Consider $w = u - \varepsilon h$ in T

$$\mathcal{L}w = \mathcal{L}u - \varepsilon \mathcal{L}h \geq 0$$

Apply weak MP to w in T, $\sup\limits_{T} w \leq \sup\limits_{\partial T} w^+ = u(x_0)$

$$0 \leq \lim_{t \to 0^+} \frac{u(x_0) - w(x_0 - te)}{t} = \lim_{t \to 0^+} \frac{u(x_0) - u(x_0 - te) + \varepsilon h(x_0 - te)}{t}$$

$$\Longrightarrow \lim_{t \to 0^+} \frac{u(x_0) - u(x_0 - te)}{t} \geq -\varepsilon \lim_{t \to 0^+} \frac{h(x_0 - te)}{t} = \varepsilon h'\Big|_{x_0} = \varepsilon \cdot \frac{\theta_2}{\lambda} > 0.$$

□

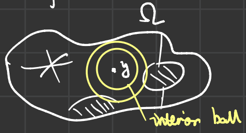
Thm (Strong Maximum Principle)

Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, $\mathcal{L}u \geq 0$

(i) If $c \leq 0$ in Ω, then $\exists x_0 \in \Omega$ s.t. $u(x_0) = \sup\limits_{\Omega} u \geq 0$. $\Longrightarrow u = \text{const}$.

(ii) If $c = 0$ in Ω. then $\exists x_0 \in \Omega$. s.t $u(x_0) = \sup\limits_{\Omega} u$ $\Longrightarrow u = \text{const}$.

Proof: Consider $\Sigma = \{x \in \Omega : u(x) = u(x_0) = \sup\limits_{\Omega} u\}$



$y \in \Omega \setminus \Sigma$ s.t $d(y, \partial\Omega) > d(y, \partial\Sigma)$

(i) ✓

(ii) Apply $v = u - \inf u \geq 0$

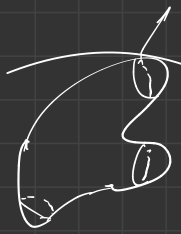
$$\underset{\text{“ } c=0 \text{ ”}}{\mathcal{L}v \overset{\uparrow}{=} \mathcal{L}u} \geq 0 \quad ✓$$

□

[Aleksandrov's soap theorem]

Let $M \subseteq \mathbb{R}^{n+1}$ be a $\underset{\text{紧致无边}}{\underline{\text{closed}}}$ embedded. smooth hyper surface.

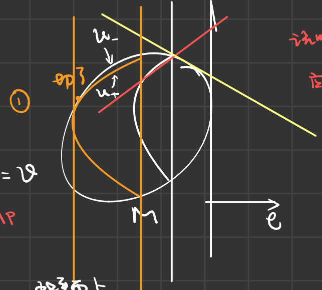
and $H_M = \text{const}$. Then M must be a sphere.



考虑球面，此处平均曲率比M小但大于0

$\therefore H_M > 0$.

Proof

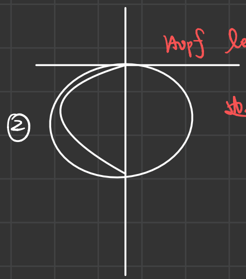


说明这个圆球
应该是轴对称,
所有方向都
这样,就是圆.

$u_+ - u_- = v$

内部 MP

两平面上
所的 所围



Hopf lemma.

边界 MP.

u_+, u_- satisfies $F_{ij}(Du)u_{ij} = H$. $\quad F_{ij}(\xi) = \frac{1}{\sqrt{1+|\xi|^2}}\left(\delta_{ij} - \frac{\xi_i\xi_j}{1+|\xi|^2}\right)$

$$F_{ij}(Du_\pm)(u_\pm)_{ij} = C$$

$$\Longrightarrow 0 = C - C = F_{ij}(Du_+)(u_+)_{ij} - F_{ij}(Du_-)(u_-)_{ij}$$

$$= \int_0^1 \frac{d}{dt}\left[F_{ij}\left(D(tu_+ + (1-t)u_-)\right)(tu_+ + (1-t)u_-)_{ij}\right]dt$$

$$0 = \underbrace{\left(\int_0^1 F_{ij}(\cdots)\,dt\right)}_{a_{ij}(x)} (\underbrace{u_+ - u_-}_{v})_{ij} + \underbrace{\left(\int_0^1 \square\right)_k}_{b_k(x)} \underbrace{D_k(u_+ - u_-)}_{v_k}$$

∵ M, C^∞ 曲面

"一致椭圆"

Apply strong mp

Thm (Aleksomolrov maximum principle) (A-B-P estimates)

Consider $\mathcal{L}u := a_{ij}u_{ij} + b_i u_i + cu \geq f$ in Ω, $u \in C^2(\Omega) \cap C(\bar{\Omega})$, Suppos

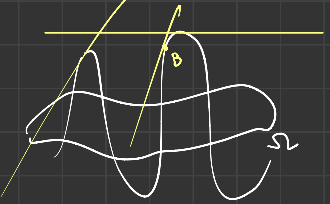
$a_{ij} > 0$ $\quad \mathcal{D} = \det a_{ij} > 0$ in Ω. $\quad c \leq 0$ in Ω $\quad \frac{|b|}{\mathcal{D}^{1/n}}$. $\frac{|f|}{\mathcal{D}^{\frac{1}{n}}} \in L^n(\Omega)$

Then $\quad \sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C\,d\left\|\frac{f^-}{\mathcal{D}^{1/n}}\right\|_{L^n(\Gamma)}$ $\quad f^- = \max\{0, -f\}$

d → diam Ω

$$C = C\left(n, \left|\frac{|b|}{\mathcal{D}^{\frac{1}{n}}}\right|_{L^n}\right)$$

$$\Gamma = \Gamma_u^+ = \{x \in \Omega.\ u(y) \leq \underbrace{u(x) + Du(x)(y-x)}_{l_x(y)},\ \forall y \in \Omega\}$$



u的图,全部在
切平面下方 (B就不行)

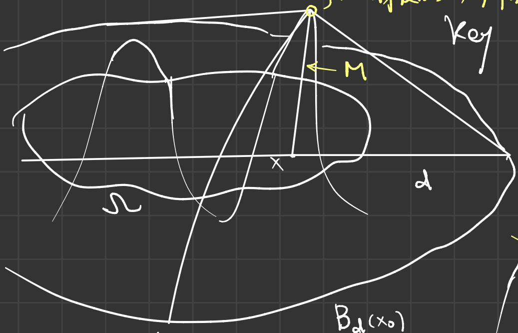
Proof: Consider $v = u - \sup_{\partial\Omega} u^+$, $v|_{\partial\Omega} \leq 0$

$$\mathcal{L}v = \mathcal{L}u - c\sup_{\partial\Omega} u^+ \geq f$$

We show $\sup_{\Omega} v \leq C\,d\left\|\frac{f^-}{\mathcal{D}^{1/n}}\right\|_{L^n(\Gamma)}$ with $u|_{\partial\Omega} \leq 0$

可微. 梯度很多, 切平面很多. 构成了集合 $B_{M/d}$

Key geometric observation



作个锥.

$\xi \in B_{\frac{M}{d}} \subseteq Du(\Gamma^+ \cap \Omega^+)$, $\Omega^+ = \{u>0\}$

解释

$u(y) - l_x(y) = 0$ 局部极大值点.

$D^2u(x) - \underbrace{D^2 l_x(y)}_{\text{linear function}} \le 0$

$D^2 \cdot = 0$

$A = (a_{ij})$

Let $g \in L^1_{loc}(\mathbb{R}^n)$. $g \geq 0$

$$\int_{B_{M/d}} g \le \int_{Du(\underbrace{\Gamma^+\cap\Omega^+}_{x})} g(y)\frac{dy}{dx}dx \le \int_{\Gamma^+\cap\Omega^+} g(Du(x)) \frac{\det(-AD^2u(x))}{\boxed{\det A}}$$

$\mathcal{D}$ —

$y = Du(x)$. $\frac{dy}{dx} = |D^2u(x)|$

$(\because D^2u(x) \le 0 \quad x \in \Gamma^+)$

$\lambda_1 \lambda_2 \cdots \lambda_n \le \left(\frac{1}{n}\Sigma\lambda_i\right)^n$

$$\Rightarrow \det(-AD^2u(x)) \le C_n(-a_{ij}u_{ij})^n$$

$$\le C_n[b_iu_i + \underbrace{cu}_{\le 0}\boxed{-f}]^n$$

$x \in \Gamma^+ \cap \Omega^+$ $\quad +f^-$

$$\le C_n(b_iu_i + f^-)^n$$

$$\le C_n\left((b, f^-/\mu)\cdot(Du, \mu)\right)^n$$

Hölder ineq $\quad \le C_n\left(|b|^n + \frac{|f^-|^n}{\mu^n}\right)(|Du|^n + \mu^n)$

$Du(\Gamma^+ \cap \Omega^+)$ 与 $\Gamma^+ \cap \Omega^+$ 微分同胚 才行.

由隐函数定理 $D^2u(x) < 0$ 才行 $\quad u_\varepsilon = u - \frac{\varepsilon}{2}|x|^2$

$Du_\varepsilon(\Gamma^+\cap\Omega^+)$ " $\approx$ " $Du(\Gamma^+\cap\Omega^+)$.

$$\le \int_{\Gamma^+\cap\Omega^+} \frac{g(Du(x))}{\mathcal{D}} \left|\frac{1}{n}(-a_{ij}u_{ij})\right|^n$$

$$\int_{B_{M/d}} g \le C_n \int_{\Gamma^+\cap\Omega^+} \frac{g(Du)}{\mathcal{D}}(|Du|^n + \mu^n)\left(|b|^n + \frac{(f^-)^n}{\mu^n}\right)$$

Take $g(\xi) = \frac{1}{\mu^n + |\xi|^n}$ (局部可积的)

$$C_n \int_0^{M/d} \frac{t^{n-1}}{\mu^n + t^n} dt \le C'_n \int_{\Gamma^+\cap\Omega^+} \frac{|b|^n}{\mathcal{D}} + \frac{(f^-)^n}{\mathcal{D}\mu^n}$$

$$\parallel$$

$$\ln\left(\frac{M^n}{d^n\mu^n} + 1\right)$$

$$\Longrightarrow \quad M^n \le \left[\exp\left(C\int_\Gamma \frac{|b|^n}{\mathcal{D}} + \frac{|f^-|^n}{\mathcal{D}\mu^n}\right) - 1\right]\mu^n d^n$$

$\frac{1}{\mu^n}\left\|\frac{(f^-)}{\mathcal{D}^{\frac{1}{n}}}\right\|^n_{L^n(\Gamma)}$

$\left\|\frac{|b|}{\mathcal{D}^{\frac{1}{n}}}\right\|^n_{L^n(\Gamma)}$

(i) if $f^- \equiv 0$ 令 $\mu \to 0$

(ii) if $\left\|\frac{f^-}{\mathcal{D}^{\frac{1}{n}}}\right\|_{L^n(\Gamma)} \neq 0$, 取 $\mu = \left\|\frac{f^-}{\mathcal{D}^{\frac{1}{n}}}\right\|_{L^n(\Gamma)}$

\#

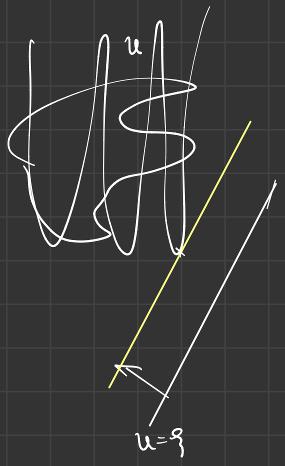
"Fake Proof" (等周不等式)

Isoperimetric inequality (X. Cabre)

$$\begin{cases} \Delta u = \frac{|\partial\Omega|}{|\Omega|} & \text{in } \Omega \\ \partial_\nu u = 1 & \text{on } \partial\Omega \end{cases} \qquad |\partial\Omega| = \int_{\partial\Omega} \partial_\nu u =$$

下接触集

$$T_u^- = \{x \in \Omega : u(y) \geq u(x) + Du(x)(y-x),\ \forall y \in \Omega$$



$\xi \in B_1(0) \subseteq Du(\Gamma)$

$$|B_1| = \int_{Du(\Gamma)} 1 = \int_\Gamma \det D^2 u \leq \int_\Gamma \boxed{\left(\frac{1}{n}\Delta u\right)^n} \to \left(\frac{1}{n}\frac{|\partial\Omega|}{|\Omega|}\right)^n$$

$$|B_1|^{\frac{1}{n}} \leq \left[\frac{1}{n}\frac{|\partial\Omega|}{|\Omega|}\right] |\Omega|^{\frac{1}{n}} = \frac{1}{n}\frac{|\partial\Omega|}{|\Omega|^{1-\frac{1}{n}}}$$

面积可以控制体积. 且仅达到最佳常数 (球面)

$$\Longrightarrow |\partial\Omega| \geq n|B_1|^{\frac{1}{n}}|\Omega|^{1-\frac{1}{n}}$$

$\star$ $a_{ij}u_{ij} = f(x)\sqrt{1+|Du|^2}$ $\quad a_{ij} = \delta_{ij} - \frac{u_i u_j}{1+|Du|^2}$ (mean curvature equation)

$$\int_{B_{r/d}} g \leq \int_{\Gamma\cap\Omega^+} g(Du)\det(-D^2u) = \int_{\Gamma\cap\Omega^+} \frac{g(Du)}{D}\det(-AD^2u) \leq C_n \int_{\Gamma\cap\Omega^+} \frac{g(Du)}{D}|f|^n (1+|Du|^2)^{n/2}$$

$$D = \det a_{ij} = \frac{1}{1+|Du|^2} \begin{pmatrix} 1 - \frac{|Du|^2}{1+|Du|^2} & 0 \\ 0 & 1 \end{pmatrix}$$

旋转总能做到 $\nearrow$

$$\begin{pmatrix} u_1 = |Du| \\ u_i = 0 \quad i \geq 2 \end{pmatrix}$$

Take $g(\xi) = \frac{1}{(1+|\xi|^2)^{\frac{n}{2}+1}}$

$$\int_0^{\frac{M}{d}} \frac{t^{n-1}}{(1+t^2)^{\frac{n}{2}+1}}\,dt \leq C_n \overset{\leq \delta_0}{\bigcirc\!\!\!\!\!\!\!\!\!\;\; \|f\|^n_{L^n(\Gamma)}} < \int_0^\infty \frac{t^{n-1}}{(1+t^2)^{\frac{n}{2}+1}}\,dt$$

$\Rightarrow$ M 有界. (在 f 不太大时).

Thm Let $u \geq 0$, $B_r(0) \subseteq \Omega$ be a solution to $(*)$

Suppose $|f|_{C^{0,1}(\Omega)} \leq C_0$. Then $|Du(0)| \leq \exp\left(\underset{n.M.C_0}{C_1} + \underset{n.C_0}{C_2}\frac{M^2}{r^2}\right)$ $\quad M = \sup_\Omega u$

Proof: X.-J. Wang (1998. Math Z)

Apply MP to $G = \eta(x)\varphi(u)\log u_\xi$, $\eta = \left(1 - \frac{|x|^2}{r^2}\right)_+$, $\varphi = 1 + \frac{u}{M}$

$\square$

Lect # 3 Schauder estimate

$\mathcal{L}u = f$ in Ω.

$d = |x-y|$

Thm let $u \in C^{2,\alpha}(\Omega)$ be a solution to $\Delta u = f$ in B_1

Then $\forall x, y \in B_{1/2}$

(X.J. Wang 2006)

$$|D^2u(x) - D^2u(y)| \leq C_n\left[d\|u\|_{L^\infty(B_1)} + \int_0^{C_0 d}\frac{\omega(r)}{r} + d\int_{C_1 d}^1 \frac{\omega(r)}{r^2}\right]$$

C_n, C_0, C_1. are uniserval const.

$$\omega(r) = \sup_{|x-y| < r}|f(x) - f(y)|$$

$f \in C^\alpha$, $\omega(r) = r^\alpha$, $\int_0^d \frac{\omega(r)}{r} = d^\alpha$

$$d\int_d^1 \frac{\omega(r)}{r^2} = d\int_d^1 r^{\alpha-2} = \square\, r^{\alpha-1}\Big|_d^1 \approx d\square + d^\alpha$$

$$|D^2u(x) - D^2u(y)| \leq C\left[d\|u\|_{L^\infty(B_1)} + Cd^\alpha + dC + d^\alpha\right]$$

$$\frac{D^2u(x) - D^2u(y)}{|x-y|^\alpha} \leq C\left[d^{1-\alpha}\|u\|_{L^\infty} + C_1\right]$$

$\Rightarrow [D^2u]_\alpha$ is bounded.

$$a_{ij} u_{ij} + b_i u_i + cu = f$$

冻固系数.

$$a_{ij}(x_0) u_{ij} = \underline{\left(a_{ij}(x_0) - a_{ij}(x)\right) u_{ij} + b_i u_i + cu - f} \searrow \hat{f} \in C^{\alpha} \text{ if.}$$

$\hat{u}(y) = u(\underset{x}{\underline{Ly}})$ 作仿射变换.

就变为
(X.J. Wang 2006)

<u>Thm</u> Let $f \in C^{\alpha}(\Omega)$ and $u \in C^{2,\alpha}(\Omega)$ be a solution to $\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$ $\searrow C^{2,\alpha}$

$(\lambda\delta_{ij} \le a_{ij} \le \Lambda\delta_{ij}, \; a_{ij}, b_i, c \in C^{\alpha}(\Omega))$ $\in C^{2,\alpha}$

For $\Omega' \subset\subset \Omega$, $\exists\, C = C(\Omega', \Omega, \alpha, n, a_{ij}, b_i, c, \lambda, \Lambda)$

s.t. $\|u\|_{C^{2,\alpha}(\Omega')} \le C\left(\|u\|_{L^{\infty}(\Omega)} + \|f\|_{C^{\alpha}(\Omega)}\right)$

考虑零边值: $v = u - \varphi$, $\begin{cases} \mathcal{L}v = \underset{f}{\underline{\mathcal{L}u}} - \underline{\underline{\mathcal{L}\varphi}} = \hat{f} \in C^{\alpha} \\ v = 0 \end{cases}$
即可

Application: (Continuity method)

<u>Thm</u> $\begin{cases} \Delta u = f \\ u = g \end{cases}$ has a unique $C^{2,\alpha}$ solution for any $f \in C^{\alpha}, \varphi \in C^{2,\alpha}$

$$\Rightarrow \begin{cases} \mathcal{L}u = f \\ u = \varphi \end{cases} \quad \text{-- -- -- -- -- -- --}$$

($C < 0$ 时. 唯一性?)

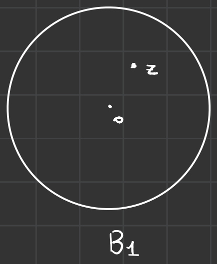
<u>Idea</u> $\mathcal{L}_t = t\mathcal{L} + (1-t)\Delta$, $I = \left\{ t \in [0,1] : \begin{cases} \mathcal{L}_t u = f \\ u = \varphi \end{cases} \exists!\ C^{2,\alpha} \text{ Solution } \forall f \in C^{\alpha}, \varphi \in C^{2,\alpha} \right\}$

$0 \in$ ↗ (非空).

前提: 全局 Schauder estimate.

$|t-s| < \delta$. 可构造

Fix t. T_t 压缩映射. $s \in I$.



$\Delta u = f$ (follow argument of X. J. Wang 2006)

Given $z \in B_{k_0+3} \setminus B_{k_0+4}$, $B_k = B_{\rho^k}(0)$ $\quad \rho = \frac{1}{2}$

($k \uparrow$ ball $\downarrow$)

We hope to estimate

$$|D^2u(z) - D^2u(0)| \leq C \left(|z| \sup_{B_1} |u| + \int_0^{C_0|z|} \frac{\omega(r)}{r} dr + |z| \int_{C_0|z|}^{1} \frac{\omega(r)}{r^2} dr \right)$$

$$\omega(r) = \sup_{|x-y|<r} |f(x) - f(y)|$$

Tool: maximum principle and interior estimates of harmonic function

Consider $\begin{cases} \Delta u_k = f(0) & \text{in } B_k \\ u_k = u & \text{on } \partial B_k \end{cases}$

1° $\quad W_\pm = u - u_k \pm \frac{\omega(\rho^k)}{2n} (\rho^{2k} - |x|^2)$ in B_k

$$\begin{cases} \Delta W_\pm = f - f(0) \mp \omega(\rho^k) & \text{in } B_k \\ W_\pm = 0 & \text{on } \partial B_k \end{cases} \qquad \begin{rcases} \Delta W_+ \leq 0 & \text{in } B_k \\ W_+ = 0 & \text{on } \partial B_k \end{rcases}$$

$W_+ \geq 0$. $W_- \leq 0$ $\Rightarrow u_k - \frac{\omega(\rho^k)}{2n}(\rho^{2k} - |x|^2) \leq u \leq u_k + \frac{\omega(\rho^k)}{2n}(\rho^{2k} - |x|^2)$

$\Rightarrow \|u - u_k\|_{\infty, B_k} \leq C \rho^{2k} \omega(\rho^k)$

$\Rightarrow \|u_{k+1} - u_k\|_{\infty, B_{k+1}} \leq \|u - u_k\|_{\infty, B_k} + \|u - u_{k+1}\|_{\infty, B_k}$

$\leq C \rho^{2k} \omega(\rho^k)$.

2° Consider $Q(x) = u(0) + x \cdot Du(0) + \frac{1}{2} x D^2u(0) x$

✓ $\Delta(u_k - Q) = 0$ $\qquad$ ✓ $\|u_k - Q\|_{\infty, B_k} \leq \underbrace{\|u_k - u\|_{\infty, B_k}}_{C\rho^{2k}\omega(\rho^k)} + \underbrace{\|u - Q\|_{\infty, B_k}}_{o(\rho^{2k})}$

By interior estimates

$$|D^2u_k(0) - D^2u(0)| \leq \frac{C}{\rho^{2k}} \left(\rho^{2k} \omega(\rho^k) + o(\rho^{2k}) \right)$$

$$\longrightarrow 0$$

$$|D^k u(0)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B_r(0))} \leq \frac{C_k}{r^k} \|u\|_{\infty, B_r}$$

$$|D^2u(z)-D^2u(0)| \le \underbrace{|D^2u_{k_0}(z)-D^2u_{k_0}(0)|}_{I_1} + \underbrace{|D^2u(0)-D^2u_{k_0}(0)|}_{I_2}$$

$$+ \underbrace{|D^2u_{k_0}(z)-D^2u(z)|}_{I_3}$$

For I_2:

$$I_2 \le \sum_{j=k_0}^{\infty} |\underbrace{D^2u_{j+1}(0)-D^2u_j(0)}_{\shortparallel\; D^2(\underline{u_{j+1}-u_j})(0)}| \le \sum_{j\ge k_0} \frac{\rho^{2j}}{(\rho^j)^2}\,\omega(\rho^j)$$

harmonic

$$\le \frac{C}{1-\rho}\sum_{j\ge k_0}\frac{\omega(\rho^j)}{\rho^{j-1}}\,\rho^{j-1}(1-\rho)$$

$$= C\sum_{j\ge k_0}\underbrace{\frac{\omega(\rho^j)}{\rho^{j-1}}(\rho^j-\rho^{j-1})}_{\shortparallel\;\int_{\rho^j}^{\rho^{j-1}}\frac{\omega(r)}{r}dr}$$

$$\le C\int_0^{C_0|z|}\frac{\omega(r)}{r}\,dr$$

For I_3, Consider v_k. $\begin{cases}\Delta v_k = f(z) & \text{in } B_k(z)\\ v_k = u & \text{on } \partial B_k(z)\end{cases}$

(Exercises)

$$I_3 \le \underbrace{|D^2u_{k_0}(z)-D^2v_{k_0}(z)|}_{?} + |D^2v_{k_0}(z)-D^2u(z)|$$

$$\Delta(u_{k_0}-v_{k_0}) = f(0)-f(z) = \Delta\left(\frac{f(0)-f(z)}{2n}(|x|^2-\rho^{2k})\right)$$

For I_1,

$$I_1 \le |D^2u_{k_0-1}(z)-D^2u_{k_0-1}(0)| + |D^2(\underbrace{u_{k_0}-u_{k_0-1}}_{h_{k_0}})(z)-D^2(u_{k_0}-u_{k_0-1})(0)|$$

$$\le |D^2u_0(z)-D^2u_0(0)| + \sum_{j=1}^{k_0}|D^2h_j(z)-D^2h_j(0)|$$

$$\le |z|\cdot|D^3u_0|_{\infty,B_{\frac12}} + C|z|\sum_{j=1}^{k_0}\boxed{|D^3h_j|_{\infty,B_{j+1}}} \le \frac{\omega(\rho^j)}{\rho(1-\rho)\rho^{j-1}\rho^{j-1}}\underbrace{\rho^{j-1}(1-\rho)}_{\rho^j-\rho^{j-1}}$$

$$\Uparrow\quad\int_{\rho^j}^{\rho^{j-1}}\frac{\omega(r)}{r^2}\,dr$$

$$I_1 \le |z|\cdot|D^3u_0|_{\infty,B_{\frac12}} + C|z|\underbrace{\sum_{j=1}^{k_0}\int_{\rho^j}^{\rho^{j-1}}\frac{\omega(r)}{r^2}\,dr}_{\int_{\rho^{k_0}}^{1}\frac{\omega(r)}{r^2}dr}$$

$$\rho^{k_0}\sim C|z|$$

$$\Delta u_0 = f(0) = \Delta\left(\frac{f(0)}{2n}(|x|^2-1)\right)$$

$$|D^3u_0| = \left|D^3\left[\underbrace{u_0-\left(\frac{f(0)}{2n}(|x|^2-1)\right)}_{\text{harmonic}}\right]\right| \le \left\|u_0-\frac{f(0)}{2n}(|x|^2-1)\right\|_\infty \le \sup_{B_1}|u|$$

$$\|D^2u\|_{C^{2,\alpha}(B_{\frac{1}{2}})} \le C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)}\right)$$

$$\Delta u = f \text{ in } \Omega, \quad \|u\|_{C^{2,\alpha}(\Omega')} \le C_{\Omega,\Omega'}\left(\|u\|_{L^\infty} + \|f\|_{C^\alpha(\Omega)}\right)$$

Thm 1. Let $\Omega \in C^{2,\alpha}$, bounded $f \in C^\alpha(\bar\Omega)$, $\phi \in C^{2,\alpha}(\bar\Omega)$. and $u \in C^{2,\alpha}(\bar\Omega)$ be a solution of $\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases}$

(uniformly elliptic + $\|a_{ij}\|_{C^\alpha}$, $\|b_i\|_{C^\alpha}$, $\|c\|_{C^\alpha} \le \Lambda$)

Then $\exists\, C = C(n, \lambda, \Lambda, \Omega)$ s.t.

$$\|u\|_{C^{2,\alpha}(\bar\Omega)} \le C\left(\|u\|_{L^\infty(\bar\Omega)} + \|\phi\|_{C^{2,\alpha}(\bar\Omega)} + \|f\|_{C^\alpha(\bar\Omega)}\right)$$

Thm Let Ω, ϕ, $\mathcal{L}$ as above, $c \le 0$ in Ω Then $\forall f \in C^\alpha(\Omega)$ $\phi \in C^{2,\alpha}(\bar\Omega)$, $\exists!\, u \in C^{2,\alpha}(\bar\Omega)$ solves

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega. \end{cases}$$

key ingradient Let Ω, $\mathcal{L}$ be as Thm and $\phi \in C^{2,\alpha}(\bar\Omega)$

If $\begin{cases} \Delta u = f & \text{in } \Omega. \\ u = \phi & \text{on } \partial\Omega \end{cases}$ has a unique $C^{2,\alpha}(\bar\Omega)$ solution for all $f \in C^\alpha(\bar\Omega)$

then $\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases}$ also has a (unique) $C^{2,\alpha}(\bar\Omega)$ solution for all such f and ϕ

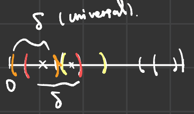
Consider a family of operator $\mathcal{L}_t = t\mathcal{L} + (1-t)\Delta$ $\quad t \in [0,1]$

$$a^t_{ij} \ge \lambda' \delta_{ij} \,/\, |a^t_{ij}|_{C^\alpha}, \ |b^t_i|_{C^\alpha}, \ |c^t|_{C^\alpha} \le \Lambda'$$

λ' & Λ' idepent of t

Define " $I = \left\{ \sigma \in [0,1] : \begin{cases} \mathcal{L}_\sigma w = f & \text{in } \Omega \\ w = \phi & \text{on } \partial\Omega \end{cases}$ has a (unique) solution for all $f \in C^{\alpha}(\bar{\Omega})$ and $\phi \in C^{2,\alpha}(\bar{\Omega})$ $\right\}$

$0 \in I$



Assume $s \in I$. We expect to show, $\exists$ universal δ s.t.

if $|s-t| \le \delta$, then $t \in I$

if $\mathcal{L}_t u = f$, $\mathcal{L}_s u = \mathcal{L}_s u + f - \mathcal{L}_t u = f + s\mathcal{L}u + (1-s)\Delta u - (t\mathcal{L}u + (1-t)\Delta u)$

$= f + (s-t)(\mathcal{L} - \Delta)u \iff u = \mathcal{L}_s^{-1}(f + (s-t)\Delta u) = T_t u$

Define $T_t : \underline{\chi = \{ u \in C^{2,\alpha}(\bar{\Omega}),\ u|_{\partial\Omega} = 0 \}} \longrightarrow \chi$

Banach space.

$u \longmapsto T_t u = \mathcal{L}_s^{-1}(f + (s-t)(\mathcal{L} - \Delta)u)$

$\mathcal{L}_s^{-1}\psi$ is <u>the</u> solution of $\begin{cases} \mathcal{L}_s w = \psi & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$ (∵ $c \le 0$. 解唯一.
又 $s \in I$. 解存在)

We show that T_t is a contraction maps (?)

$$\|T_t u - T_t v\|_{C^{2,\alpha}} = \|\mathcal{L}_s^{-1}((s-t)(\mathcal{L}-\Delta)(u-v))\|_{C^{2,\alpha}}$$

$$= |s-t| \, \|\mathcal{L}_s^{-1}((\mathcal{L}-\Delta)(u-v))\|_{C^{2,\alpha}}$$

$$\le C|s-t| \, \|(\mathcal{L}-\Delta)(u-v)\|_{C^{\alpha}} \quad \text{(maximal principle \& Schauder)}$$

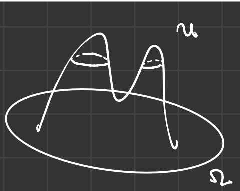
$$\le \underbrace{C'|s-t|}_{\le \frac{1}{2}} \|u-v\|_{C^{2,\alpha}}$$

$\swarrow$

$\exists u$, s.t. $T_t u = u$.

#

$E(u) = \int_\Omega F(\nabla u)$, $F(q) = \sqrt{1+|q|^2}$



$$0 = \frac{d}{dt} E[u+t\varphi] = \int_\Omega \frac{d}{dt} F(D(u+t\varphi))$$

$\nearrow$ $\forall \varphi \in C_0^\infty(\Omega)$

$$= \int_\Omega F_{q_i} \varphi_i = -\int_\Omega \partial_i [F_{q_i}(Du)] \varphi$$

$$\Rightarrow 0 = \partial_i (F_{q_i}(Du)) = F_{q_i q_j} D_{ij}^2 u, \qquad \lambda \delta_{ij} \le F_{q_i q_j}(Du) \le \Lambda \delta_{ij}$$

Take $a_{ij}(x) = F_{q_i q_j}(Du(x)) \in C^{\alpha} \xrightarrow[\text{Schauder}]{} u \in C^{2,\alpha}$ $\quad u \in C^{0,1}(\Omega)$.

$\nearrow$ $u \in C^{1,\alpha}$ $\quad \xrightarrow[\text{bootstrapping.}]{} u \in C^\infty$

$$\partial_i\left(F_{q_i q_j}\,\partial_j u_e\right)=0$$

$\Omega \longrightarrow u_e$ satisfies $\quad \partial_i(a_{ij}\partial_j v)=0, \quad a_{ij}=F_{q_i q_j}(Du)$

$$\lambda\delta_{ij} \le a_{ij} \le \Lambda\delta_{ij} \quad \Longrightarrow \quad \underline{v \in \mathcal{C}^{\alpha}.}$$

De Giorgi – Moser (1960s)

<u>Thm</u> Let $u \in H^1(B_1)$ be a solution of $\partial_i(a_{ij}\partial_j u)=f$

(i) $\lambda\delta_{ij} \le a_{ij} \le \Lambda\delta_{ij}$, (ii) $f \in L^q(B_1)$, $q > n/2$. Then

$$|u|_{C^{\alpha}(B_{\frac12})} \le \underset{\overset{\swarrow}{n.\ \lambda.\ \Lambda.\ q}}{C}\left(|u|_{L^2(B_1)} + |f|_{L^q(B_1)}\right)$$

Local boundedness (From L^2 to L^∞ bound)

<u>prop 1</u> If $\partial_i(a_{ij}u_j) \ge f$ in B_1. $f \in L^q(B_1)$, $q > n/2$. Then

$$\sup_{B_{\frac12}} u^+ \le C\left(|u^+|_{L^2(B_1)} + |f|_{L^q(B_1)}\right)$$

<u>Def</u> $\partial_i(a_{ij}\partial_j u) \ge f$ in B_1 means $\forall\ \varphi \in H^1_0(B_1)$ and $\varphi \ge 0$. We have

$$-\int a_{ij}\varphi_i u_j = \int \varphi\,\partial_i(a_{ij}\partial_j u) \ge \int \varphi f$$

$$\Rightarrow \int a_{ij}\varphi_i u_j \le -\int_{B_1}\varphi f$$

<u>Sobolev (Evans, chapter 5)</u>

$$|w|_{L^2(B_1)} \lesssim |w|_{L^{2^*}(B_1)} \le C_{2^*}|Dw|_{L^2(B_1)}, \qquad 2^* = \begin{cases} \frac{2n}{n-2}\ (>2) & n \ge 3 \\ \text{any } p > 2 & n=2 \end{cases}$$

$$\left(W^{k.p} \hookrightarrow L^{\frac{np}{n-kp}}\right)$$

<u>Energy estimates.</u>

Lemma If $\overset{u \ge 0}{{}^{\vee}}\partial_i(a_{ij}\partial_j u) \ge f \in L^q(B_1)$, $q > n/2$, then $\forall\ \overset{\nearrow \text{cut off func}}{\eta} \in H^1_0(B_1)$

$$\int_{B_1}|D(\eta u)|^2 \le C\sup_{B_1}|D\eta|^2\int_{\operatorname{Supp}\eta}u^2 + C\sup_{B_1}\eta^2\,|f|^2_{L^q}\,|\operatorname{supp}(\eta u)|^{1-\frac{1}{q}}$$

Proof: Take $g = u\eta^2$ in Definition

$$\int_{B_1} a_{ij}\,\partial_i(u\eta^2)\,u_j \le -\int_{B_1} f u\eta^2$$

$$\text{l.h.s.} = \int_{B_1} a_{ij}\left(\eta\,\partial_i(u\eta) + u\eta\,\partial_i\eta\right)u_j = \int_{B_1} a_{ij}\,\partial_i(u\eta)\left[\partial_j(u\eta) - u\,\partial_j\eta\right] + \int a_{ij}\,u\eta\,\eta_i u_j$$

$$\geqslant \lambda\int_{B_1}|D(u\eta)|^2 - \int a_{ij}\,(\cancel{u_i\eta} + u\eta_i)\,u\,\eta_j$$

$$+ \cancel{\int a_{ij}\,u\eta\,\eta_i u_j}$$

$$\geqslant \lambda\int_{B_1}|D(\eta u)|^2 - \Lambda\sup_{B_1}|D\eta|^2\int_{\operatorname{supp}\eta} u^2$$

$$\text{r.h.s} \leqslant \int|f||u\eta||\eta| \underset{\substack{\uparrow\\ \text{Hölder}}}{\le} \sup_{B_1}|\eta|\,|f|_{L^q}\,\boxed{|u\eta|_{L^{q^*}}}$$

$$\le |u\eta|_{2^*}\,|\operatorname{supp}(u\eta)|^{\frac{1}{q^*}-\frac{1}{2^*}} \qquad 1-\frac{1}{q}$$

$$\le \frac{1}{\delta}\sup_{B_1}\eta^2\,|f|^2_{L^q}\,|\operatorname{supp}(u\eta)|^{2-\frac{2}{q}-\frac{2}{2^*}} + \delta\,\underline{|u\eta|^2_{2^*}} \le C\int|D(\eta u)|^2.$$

$$\delta C \le \frac{\lambda}{2} \quad (\text{取}\delta\text{足够小})$$

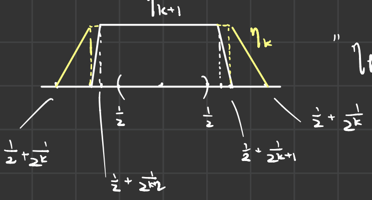
$$2 - \frac{2}{q} - \frac{2}{2^*} = 1 - \frac{1}{q} + \underbrace{\left[1 - \frac{1}{q} - \frac{2}{2^*}\right]}_{>0}$$

□

Remark: if $\partial_i(a_{ij}\partial_j u) \geqslant f$, then $\partial_i(a_{ij}\partial_j u^+) \geqslant -|f|$

proof of prop 1

1° cut-offs. $\eta_k \geqslant \eta_{k+1} \longrightarrow 1_{B_{\frac12}}$,
$$\eta_k = \begin{cases} 1 & B_{\frac12+\frac{1}{2^{k+1}}} \\ |D\eta_k| \lesssim 2^k & B_{\frac12+\frac{1}{2^k}} \setminus B_{\frac12+\frac{1}{2^{k+1}}} \\ 0 & \text{outside } B_{\frac12+\frac{1}{2^k}} \end{cases}$$



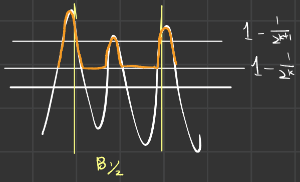
" $\eta_k = 1$ on $\operatorname{supp}\eta_{k+1}$ "

$$\hat{u} = \frac{u}{\varepsilon + \delta_0^{-1}(\|u^+\|_{L^2} + |f|_{L^q})}, \quad |\hat{u}^+|_{L^2} = \frac{|u^+|_{L^2}}{\varepsilon + \delta_0^{-1}(|u|^+_{L^2} + |f|_{L^q})} \le \delta_0$$

$$\mathcal{L}\hat{u} = \frac{f}{\varepsilon + \delta_0^{-1}(\|u^+\|_{L^2} + |f|_{L^q})} =: \hat{f} \qquad |\hat{f}|_{L^q} \le \delta_0$$

转化为 只需证. $\exists$ universal δ_0 s.t. $\|\hat{u}^+\|_{L^2}, \|\hat{f}\|_{L^q} \leq \delta_0 \Rightarrow \sup\limits_{B_{1/2}} \hat{u}^+ \leq 1$

2° Truncations. $u_k = [u - (1 - \frac{1}{2^k})]^+$



Consider $\eta_k u_k$

(i) $\eta_k u_k \geq \eta_{k+1} u_{k+1} \longrightarrow [u-1]^+ 1_{B_{1/2}}$

(ii) $\eta_k u_k > \frac{1}{2^{k+1}}$ on $\operatorname{supp}(\eta_{k+1} u_{k+1})$

3° Iteration. Consider $V_k = \int \eta_k^2 u_k^2$

$$V_k = |\eta_k u_k|^2_{L^2} \overset{\text{Hölder}}{\leq} |\eta_k u_k|^2_{L^{2^*}} |\operatorname{supp} \eta_k u_k|^{1-\frac{2}{2^*}}$$

$$\overset{\text{Sobolev}}{\leq} C \int |D(\eta_k u_k)|^2 |\operatorname{supp} \eta_k u_k|^{1-\frac{2}{2^*}}$$

$$\overset{\text{Energy estimates.}}{\leq} C\, 2^{2k} \int_{\operatorname{supp} \eta_k} u_k^2 \, |\operatorname{supp} \eta_k u_k|^{1-\frac{2}{2^*}} + C |f|^2_{L^q} |\operatorname{supp} \eta_k u_k|^{2-\frac{1}{q}-\frac{2}{2^*}}$$

$$V_{k+1} \leq C 2^{2k} \underbrace{\int \eta_k^2 u_k^2}_{v_k} \underline{|\operatorname{supp} \eta_{k+1} u_{k+1}|}^{1-\frac{2}{2^*}} + C |f|^2_{L^q} \underline{|\operatorname{supp} \eta_{k+1} u_{k+1}|}^{2-\frac{1}{q}-\frac{2}{2^*}}$$

$$v_k \geq \int_{\operatorname{supp}(\eta_{k+1} u_{k+1})} \eta_k^2 u_k^2 \geq 2^{-(k+1)} |\operatorname{supp}(\eta_{k+1} u_{k+1})| \rightsquigarrow \underline{|\sim| \leq 2^{k+1} v_k}$$

$$v_{k+1} \leq C\, 2^{2k} v_k \underline{|\operatorname{supp} \eta_{k+1} u_{k+1}|}^{\overset{\varepsilon+\frac{1}{q}=}{\boxed{1-\frac{2}{2^*}}}} + C (2^k v_k)^{\underline{2-\frac{1}{q}-\frac{1}{2^*}}} = 1+\varepsilon. \text{ 其中 } \varepsilon = 1 - \frac{1}{q} - \frac{2}{2^*} > 0$$

$|\sim|^{\frac{1}{q}}$ 为 B_1 上 measure. 被到 C中.

$$1 - \frac{1}{q} - 2\,\frac{n-2}{2n} = \frac{2}{n} - \frac{1}{q} > 0$$

$$\Longrightarrow v_{k+1} \leq C (2^{2k} v_k)^{\varepsilon+1} + C (2^{2k} v_k)^{1+\varepsilon}$$

$$\leq C\, \underline{2^{2k(1+\varepsilon)}} v_k^{1+\varepsilon}$$

$\parallel$ a^k. 其中 $a = 2^{2(1+\varepsilon)}$

$$V_{k+1} \leq C a^k V_k^{1+\varepsilon} \leq C a^k (C a^{k-1} v_{k-1}^{1+\varepsilon})^{1+\varepsilon}.$$

$$= C^{1+(1+\varepsilon)} a^{k+(k-1)(1+\varepsilon)} v_{k-1}^{(1+\varepsilon)^2}$$

$$\leq C^{\sum_{j=0}^{k} (1+\varepsilon)^j} a^{\sum_{j=0}^{k} j (1+\varepsilon)^{k-j}} V_0^{(1+\varepsilon)^{k+1}}$$

$$= \Big[\underbrace{C^{\sum_{j=0}^{k} (1+\varepsilon)^{\underset{<0}{\underline{j-(k-1)}}}} \; a^{\sum_{j=0}^{k} j (1+\varepsilon)^{\underset{<0}{\underline{-j-1}}}}}_{\text{convergence.}} \delta_0 \Big]^{(1+\varepsilon)^{k+1}}$$

$\therefore$ limit are bounded.

$V_0 = \int \eta_0^2 u^2$

$\leq \|u^+\|_{L^2(B_1)}$

$\leq \delta_0$

取 δ_0 (C. a's limit). 足够小.

$$\leq \left(\frac{1}{2}\right)^{(1+\varepsilon)^{k+1}} \longrightarrow 0$$

$\eta_k u_k \longrightarrow (u-1)^+ 1_{B_{\frac{1}{2}}}$ $\quad \therefore \sup\limits_{B_{1/2}} u^+ \leq 1$

Thm $\mathcal{L}u \triangleq \partial_i(a_{ij}\partial_j u) = f \in L^q(B_1)$, $q > n/2$. $u \in C^\alpha(B_1)$. $q > n/2$

$u \in C^\alpha(B_{1/2})$. $\forall\, x.y \in B_{1/2}$. $|u(x) - u(y)| \leq C|x-y|^\alpha \left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^q(B_1)} \right)$

We show $|u(x) - u(0)| \leq C|x|^\alpha \left(\|u\|_{L^2(B_1)} + \|f\|_{L^q(B_1)} \right)$

Prop (Osallatine. decay) $\exists$ universal $\sigma \in (0.1)$ and $c>0$. s.t.

$$\operatorname{osc}_{B_{1/2}} u \leq \sigma \operatorname{osc}_{B_1} u + C\|f\|_{L^q(B_1)}.$$

proof of Hölder estimates

$B_{1/2^k}$

Consider $\hat{u}(y) = u\left(\boxed{y/2^k}\right)$. $y \in B_1$.

$\mathcal{L}\hat{u}(y) = 4^k f(y/2^k) \triangleq \hat{f}(y)$ in B_1

Apply Osallotia. decay to $\hat{u}$ in B_1.

$$\operatorname{osc}_{B_{1/2}} \hat{u} \leq \sigma \operatorname{osc}_{B_1} \hat{u} + C\|\hat{f}\|_{L^q(B_1)} = \left(4^{-kq} \int f^q(y/2^k)\, dy\right)^{\frac{1}{q}}$$

$$\Rightarrow \operatorname{osc}_{B_{1/2}} u \leq \sigma \operatorname{osc}_{B_{1/2}} u$$

$$+ Ca^k\|f\|_{L^q(B_1)}$$

$$\leq \sigma\left(\operatorname{osc}_{B_{1/2^{k-1}}} u + C a^{k-1}\|f\|_{L^q}\right) + Ca^k\|f\|_{L^q}$$

$$= \sigma^2 \operatorname{osc}_{B_{1/2^{k-1}}} u + C\|f\|_{L^q}\left(\sigma a^{k-1} + a^k\right)$$

$$\leq \sigma^k \operatorname{osc}_{B_{1/2}} u + C\|f\|_{L^q}\left(\sum_{j=0}^{k-1} \sigma^j a^{k-j}\right)$$

$$a^k\|f\|_{L^q(B_1)} = 4^{-k}\left(2^{nk}\int f^q(y/2^k)\, dy / 2^q\right)^{\frac{1}{q}}$$

$$\leq 4^{-k}.\, 4^{kn/2q}\|f\|_{L^q(B_1)}$$

$$4^{k(1-\frac{n}{2q})} = 2 > 0$$

$$a = 4^{-\varepsilon_0} < 1$$

Let $\tau = \max\{\sigma.a\}$

$$\leq \tau^k C\left(\|u\|_{L^2(B_1)} + \|f\|_{L^q(B_1)}\right) + C\|f\|_{L^q}\boxed{\tau^k}k$$

$// \; |x|^2(-\log|x|)$

Suppose $x \in B_{1/2^k} \setminus B_{1/2^{k+1}}$ $\quad |x| \approx 2^{-k}$. $\underline{k \text{ large}}$

$$\tau^k = 2^{\log_2 \tau^k} = 2^{-k|\log_2 \tau|} \cong |x|^{|\log_2 \tau|}$$

Proof of oscillation decay

Consider $\hat u(x) = \frac{\partial u(x)}{\mathrm{osc}_{B_1(x)} u}$ $\tilde u(x) = \hat u(x) - \frac12 (\sup_{B_1} \hat u + \inf_{B_1} \hat u)$

$\rightsquigarrow \mathrm{osc}_{B_1} \tilde u = 2.$ $\sup_{B_1} \tilde u = 1.$ $\inf_{B_1} \tilde u = -1$

$\mathcal L \tilde u = \frac{2f}{\mathrm{osc}_{B_1} u} = \tilde f$

Let δ be a universal cont (to be determined)

(i) $\|\tilde f\|_{L^q} > \delta$ ($2\|f\|_{L^q} > \mathrm{osc}_{B_1} u$, $\mathrm{osc}_{B_{1/2}} \le \frac12 \mathrm{osc}_{B_1} u + \underbrace{\frac12 \mathrm{osc}_{B_1} u}_{\le \frac{1}{\delta}\|f\|_{L^2}}$

(ii) $\|\tilde f\|_{L^q} \le \delta$ either $\underline{|\{\tilde u^+ = 0\} \cap B_{3/4}| \ge \frac12 |B_{3/4}|}$ $\xrightarrow{\text{lem}}$ $\tilde u^+ \le 1-\gamma$

or $|\{\tilde u^- = 0\} \cap B_{3/4}| \ge \frac12 |B_{3/4}|$ $\frac{\boxed{2}\, \mathrm{osc}_{B_{1/2}} u}{\mathrm{osc}_{B_1} u} = \mathrm{osc}_{B_{1/2}} \tilde u \le (2-\gamma)/2 = \frac{1-\gamma}{\delta}$

Lemma 1 Let $0 \le u \le 1$, $\mathcal L u = f$ Assume

$$|\{u=0\} \cap B_{3/4}| \ge (1-\mu)\, |B_{3/4}|. \quad \text{for some } \mu \in (0, 1)$$

Then $\exists\ \delta = \delta_\mu$ $\gamma = \gamma_\mu$ st $\sup_{B_{1/2}} u \le 1 - \gamma_\mu$ provided $\|f\|_{L^q} \le \delta_\mu$

Recall: $\sup_{B_{1/2}} u \le C\left(\|u\|_{L^2(B_{\underset{\downarrow 3/4}{\boxed{1}}})} + \|f\|_{L^q(B_1)}\right)$

$\varepsilon > 0$, $|\eta_k u_k|_2 \le |\mathrm{supp}\, \eta_k u_k|^{\frac12 - \frac{1}{2^*}}$

$|\eta_k u_k|_{2^*} \le |D(\eta_k u_k)|_2$

$\le |\eta_k u_k|_2$

Consider $u_k = \left[u - (1 - \frac{1}{2^k})\right]_+$

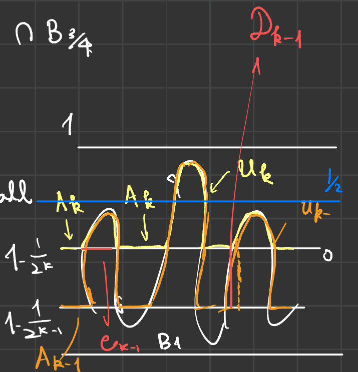
$\hat u_k = 2^k u_k$. $0 \le \hat u_k \le 1$

Define $\mathcal A_k = \{\hat u_k = 0\} \cap B_{3/4}$, $\mathcal C_k = \{\hat u_k \ge \frac12\} \cap B_{3/4}$

$\mathcal D_k = \{0 < \hat u_k < 1/2\} \cap B_{3/4}$

We aim to show $\exists\ k_0$ (finite) s.t.

$|\mathcal A_{k_0}| \approx |B_{3/4}| \xRightarrow[\substack{L^\infty \text{ estimate} \\ \text{to } \hat u_k \text{ in } B_{3/4}}]{} \sup_{B_{1/2}} \hat u_{k_0} \le q_0$ ↙ small



Transform this back to u . we get the conclusion

Pf : Lem 1

$\exists\, k = k_0$ s.t $|\mathcal{A}_k| \geq (1-\delta_0)|B_{3/4}|$. for a δ_0

$$\sup_{B_{1/2}} \hat{u}_{k_0} \leq \underbrace{C\,\delta_0^{\frac{1}{2}}|B_{3/4}|^{\frac{1}{2}}}_{<\frac{1}{4}\text{ (take }\delta_0\text{ small)}} + \underbrace{4^{k_0}\|f\|_{L^q(B_1)}}_{<\frac{1}{4}} < 1/2$$

$$\Longrightarrow \sup_{B_{1/2}} u \leq 1 - \frac{1}{2^{k_0+1}}$$

Assume NOT. $(1-\mu)|B_{3/4}| \leq |\mathcal{A}_0| \leq |\mathcal{A}_k| \leq (1-\delta_0)|B_{3/4}| \quad \forall k$

Lemma 2 Let $w \in H^1(B_r)$, $0 \leq w \leq c$ For $c > 0$. let

$$\mathcal{A} = \{w = 0\},\quad \mathcal{C} = \{w = c\}.\quad \mathcal{D} = \{0 < w < c\}.$$

Then $\exists\, C_n > 0$, s.t $|\mathcal{D}| \int_{B_r} |Dw|^2 \geq \frac{c^2 C_n}{r^{2n}} |\mathcal{C}|^{2-\frac{2}{n}} |\mathcal{A}|^2$

Apply lemma 2 to $\mathcal{A}_k$. $\mathcal{C}_k$. $\mathcal{D}_k$

$$|\mathcal{D}_k| \geq C_\mu \, |\mathcal{C}_k|^{2-\frac{2}{n}} \Big/ \boxed{\int_{B_{3/4}} |D\hat{u}_k|^2} \rightsquigarrow \left(1 + 4^k \|f\|_{L^q}^2\right)$$

$$\Longrightarrow |\mathcal{A}_k| = |\mathcal{A}_{k-1}| + |\mathcal{D}_{k-1}|$$

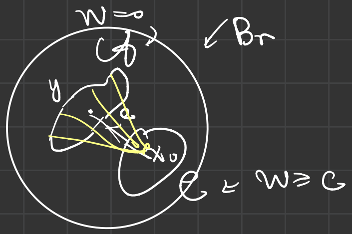
$$\geq |\mathcal{A}_0| + \sum_{j=0}^{k-1} \boxed{|\mathcal{D}_j|} \geq \frac{C_\mu \delta_0^{2-\frac{2}{n}}}{1 + 4^k \|f\|_{L^q}^2}$$

$$\geq (1-\mu)|B_{3/4}|.$$

$$|\mathcal{A}_k| + \underbrace{|\mathcal{C}_{k-1}|}_{\geq \delta_0 |B_{3/4}|} = |B_{3/4}|$$

□.

Proof of lem 2



$$\begin{aligned} C &= W(x_0) - W(y) \\ &= -\int_0^{\rho_y} \frac{d}{dt} W(x_0 - te_y)\,dt \\ &\leq \int_0^\infty |DW(x_0 - te_y)|\,dt \end{aligned}$$

Define $\Theta_{x_0,A} = \left\{ e \in S^{n-1},\ e = \frac{y-x_0}{|y-x_0|},\ y \in A \right\}$

$$\begin{array}{ll} C|\Theta_{x_0,A}| & \leq c\int_{\Theta_{x_0,A}} \int_0^\infty |DW(x_0 - te_y)|\,dt\,de \\ \rotatebox{-90}{$\geq$} & \\ \dfrac{C_n C|A|}{r^n} & \leq c\int_D \dfrac{DW(z)}{|z-x_0|^{n-1}}\,dz \end{array}$$

$$|A| \geq \int_{\Theta_{x_0,A}} \int_0^{2r} t^{n-1}\,dt\,de \leq C r^n |\Theta_{x_0,A}|$$

$$\Longrightarrow |A| \leq \frac{C_n r^n}{C} \int_D \frac{|DW(z)|}{|z-x_0|^{n-1}}\,dz$$

$$|A||e| \leq \frac{C_n r^n}{C} \int_D \int_e \frac{|DW(z)|}{|z-x_0|^{n-1}}\,dx_0\,dz \leq \frac{C_n r^n}{C} \int_D |DW(z)|$$

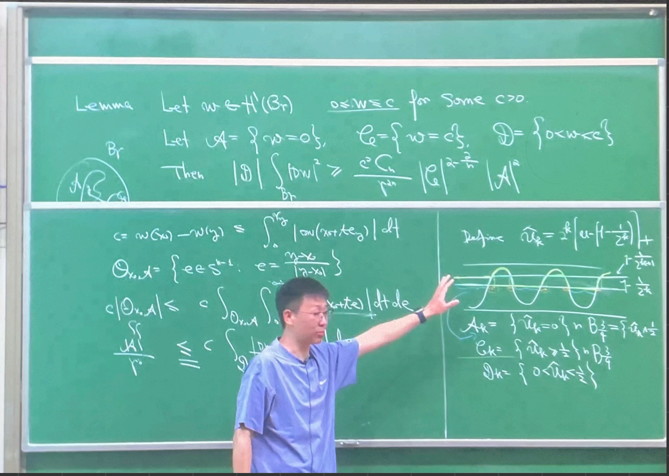
$$\int_{B_\rho(z)} \frac{dx_0}{|z-x_0|^{n-1}}\,dz$$

$$\underbrace{|z-x_0|}_{\rho^{n-1}}$$

Take $B_\rho(z)$. s.t. $|B_\rho(z)| = |e|$

$$\begin{aligned} \Longrightarrow |A||e|^{1-\frac{1}{n}} &\leq \frac{C_n r^n}{C} \int_D |DW|\,dz \\ &\leq \frac{C_n r^n}{C} \left(\int |DW|^2 \right)^{\frac{1}{2}} |D|^{\frac{1}{2}} \end{aligned}$$

□



Lemma Let $w \in H^1(B_r)$ $\underline{0 \le w \le c}$ for some $c>0$.

Let $\mathcal{A}=\{w=0\}$, $\mathcal{C}=\{w=c\}$, $\mathcal{D}=\{0<w<c\}$

Then $|\mathcal{D}| \int_{B_r} |Dw|^2 \geq \frac{c^2 C_n}{r^{2n}} |\mathcal{C}|^{2-\frac{2}{n}} |\mathcal{A}|^2$

$c = w(x_0) - w(y) \le \int_0^{r_y} |Dw(x_0+te_y)|\, dt$

$\Theta_{x_0,\mathcal{A}} = \{e \in S^{n-1} : e = \frac{y-x_0}{|y-x_0|}\}$

$c|\Theta_{x_0,\mathcal{A}}| \le c \int_{\Theta_{x_0,\mathcal{A}}} \int_0 \ldots (x_0+te)|\,dt\,de$

$\frac{|\mathcal{A}|}{r^n} \le c \int_{\mathcal{D}} \ldots$

Define $\hat{u}_k = 2^k\left[u-\left(1-\frac{1}{2^k}\right)\right]_+$

$A_k = \{\hat{u}_k = 0\} \cap B_{\frac{3}{4}} = \{\hat{u}_k \wedge \frac{1}{2}$

$C_k = \{\hat{u}_k \geq \frac{1}{2}\} \cap B_{\frac{3}{4}}$

$D_k = \{0 < \hat{u}_k < \frac{1}{2}\}$

"作了些更正"

（附李老师帅照一张. doge）

Lect # 5 Krylov-Safanov's Harnack inequality

$\underline{\text{Thm}}$ Let $u \in C(B_1)$. $Lu \triangleq a_{ij} u_{ij} = f \in C(B_1)$, bounded.

Suppose $u \geq 0$. Then $\sup\limits_{B_{1/2}} u \leq C\left(\inf\limits_{B_{1/2}} u + \|f\|_{L^n(B_1)}\right)$

$\underline{\text{Corollary}}$. $u \in C^\alpha(B_{1/2})$, $\forall x, y \in B_{1/2}$. $|u(x)-u(y)| \leq C|x-y|^\alpha \left(\|u\|_{L^\infty} + \|f\|_{L^n}\right)$

$\underline{\text{proof}}$: Consider $u - \inf\limits_{B_1} u$ / $\sup\limits_{B_1} u - u$

$$\sup_{B_{1/2}} u - \inf_{B_1} u \leq C\left(\inf_{B_{1/2}} u - \inf_{B_1} u + \|f\|_{L^n(B_1)}\right)$$

$$\sup_{B_1} u - \inf_{B_{1/2}} u \leq C\left(\sup_{B_1} u - \sup_{B_{1/2}} u + \|f\|_{L^n(B_1)}\right)$$

$$\operatorname{osc}_{B_1} u + \operatorname{osc}_{B_{1/2}} u \leq C\left(\operatorname{osc}_{B_1} u - \operatorname{osc}_{B_{1/2}} u + 2\|f\|_{L^n(B_1)}\right)$$

$$\Longrightarrow \quad \operatorname{osc}_{B_{1/2}} u \leq \underbrace{\boxed{\frac{C-1}{C+1}}}_{\sigma \in (0,1)} \operatorname{osc}_{B_1} u + \frac{2C}{C+1}\|f\|_{L^n}$$

□

interation ...

$$\hat{u} = \frac{\sup_{B_{1/2}} u}{\varepsilon + \inf\limits_{B_{1/2}} u + \delta^{-1}\|f\|_{L^n}} \leq K_0 \quad \mathcal{L}\hat{u} = \frac{f}{\sim} = \hat{f}. \quad \|\hat{f}\|_{L^n} \leq \delta$$

$$\inf \hat{u} \leq 1$$

Thm Let $u \in C(\Omega)$, $u \geqslant 0$. $\inf_{B_{1/2}} u \leq 1$. $\mathcal{L}u = f \in C(B_1)$ bounded.

Then $\exists$ universal δ_0, K_0 s.t. $\sup_{B_{1/2}} u \leq K_0$. provided $\|f\|_{L^n} \leq \delta_0$

Thm' Let $\overset{\in C(\Omega)}{u} \geqslant 0$ in $Q_{4\sqrt{n}}$, $\inf_{Q_{1/4}} u \leq 1$. $\mathcal{L}u = f \in C(Q_{4\sqrt{n}})$ bounded

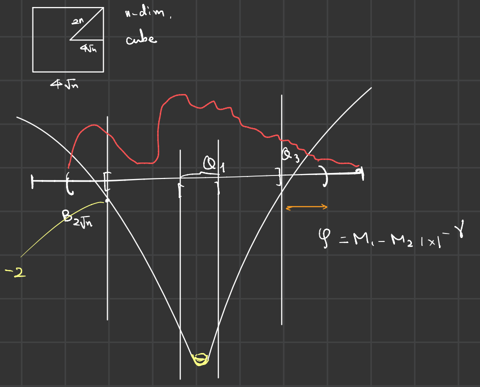
Then $\exists$ universal δ_0. $K_0 > 0$ s.t. $\sup_{Q_{1/4}} u \leq K_0$. provided $\|f\|_{L^n} \leq \delta_0$

Def. We say $u \in C(\Omega)$ is a viscosity sub/super-solution denoted by $\mathcal{L}u \geqslant f$ $(\leq)$

means $\forall x_0 \in \Omega$. $\forall \varphi \in C^2(\Omega)$. if $u - \varphi$ attain local maximum/minimum at x_0

then $\mathcal{L}\varphi(x_0) \geqslant f(x_0)$ $(\leq)$ / Crandall - Ishii - Lions.

(粘性解: 唯一性., 正則化)



- $\mathcal{L}\varphi \leq cg$ $\quad 0 \leq g \leq 1$, $\operatorname{supp} g \subseteq Q_1$
- $\varphi \leq -2$ in Q_3
- $\varphi \geqslant -M_0$
- $\varphi \geqslant 0$ in $\mathbb{R}^n \setminus B_{2\sqrt{n}}$

Consider $u + \varphi$ | $\inf_{Q_3}(u+\varphi) \leq \inf_{Q_{1/4}} u - 2 \leq -1$

ABP - estimates

$-(u+\varphi)$

?

$\mathcal{L}v \geqslant f$, $\sup_{\Omega} v \leq \sup_{\partial\Omega} v^+ + c\left(\int_{\Gamma^+ \cap \{v \geqslant 0\}} |f|^n\right)^{\frac{1}{n}}$

$$1 \leq c\left[\int_{\Gamma_- \cap \{u+\varphi \leq 0\}} (f + cg)^n\right]^{\frac{1}{n}} \leq \boxed{c\|f\|_{L^n}} + |\{\underbrace{\overset{\geqslant -M_0}{u+\varphi} \leq 0}_{u \leq M_0}\} \cap Q_1|^{1/n}$$

$$\Longrightarrow \exists \mu_0 = \mu_0(\delta_0) \quad \boxed{|\{u \leq M_0\} \cap Q_1| \geqslant \mu_0 |Q_1|.}$$

$\Leftarrow$ $k = 1$ inf.

Prop: $u \geqslant 0$ in $Q_{4\sqrt{n}}$ $\inf_{Q_3} u \leq 1$. $\mathcal{L}u = f$ in $Q_{4\sqrt{n}}$, $\exists$ universal δ_0, μ_0. M_0

s.t. if $\|f\|_{L^n(Q_{4\sqrt{n}})} \leq \delta_0$, then $|\{u > M_0^k\} \cap Q_1| \leq (1 - \mu_0)^k |Q_1|$ $\forall k$

$\Longrightarrow$ $|\{u > t\} \cap Q_1\}| \leq C A^{-\varepsilon_0}$ (weak L^{ε_0}-estimate) $-|k| |\log(1-\mu_0)|$

proof $k=1$ ✓. By induction argument $k-1$ ✓ $|\{u>M_0^{k-1}\}\cap Q_1| \le (1-\mu_0)^{k-1}|Q_1|$

We next show it holds for $\underline{\underline{k}}$

Consider $A=\{u>M_0^k\}\cap Q_1$. $B=\{u>M_0^{k-1}\}\cap Q_1$

Lemma (CZ decomposition) $A.B\subseteq Q_1$ satisfies two conditions

(i) $|A|\le\theta$ (ii) if $|A\cap Q|>\theta|Q|$. then $\tilde{Q}\subseteq B$

Then $|A|\le\theta|B|$

proof:

二进制分体

① $\{Q^i\}$, $|A\cap Q^i|>\theta|Q^i|$

② $\{\tilde{Q}^j\}$ $\tilde{Q}^j$ is 上一代 of Q^i

↑ disjoint

$A\subseteq\bigcup_i Q^i$ (except a null set)

$$|A|=|A\cap(\cup Q^i)|=|\cup(A\cap Q^i)|\le\sum_j|A\cap\tilde{Q}^j|$$

∵如果 ">" 则 不会有下一代 $\underline{Q^i}$

$$\le\theta\sum_j|\tilde{Q}_j|\le\theta B.\ \square$$

We verify $A=\{u>M_0^k\}\cap Q_1$. $B=\{u>M_0^{k-1}\}\cap Q_1$. satisfies CZ lemma.

only need to

$\boxed{\theta=1-\mu_0}$ (i) $|A|\le|\{u>M_0\}\cap Q_1|\le 1-\mu_0=\theta$ ✓

(ii) $\exists\, Q_{2^{-i}}(x_0)\subseteq Q_1$ s.t. $|A\cap Q_{2^{-i}}(x_0)|>(1-\mu_0)|Q_{2^{-i}}|$. we expect to

show $\tilde{Q}_{2^{-i}}(x_0)=Q_{2^{-(i-1)}}(x_0')\subseteq B$

By contradiction argument. If NOT

$\exists\,\tilde{x}\in\tilde{Q}_{2^{-i}}(x_0)$ s.t. $u(\tilde{x})\le M_0^{k-1}$

Consider $x=2^i(y-x_0)$ $y\in Q_{4\sqrt{n}}$ ($\Rightarrow x\in Q_{2^{-i}\cdot4\sqrt{n}}(x_0)$)

- $\tilde{u}(y)=\dfrac{u(x(y))}{M_0^{k-1}}$ in $Q_{4\sqrt{n}}$

- $\tilde{u}(y) \geq 0$;
- $\inf_{Q_3} \tilde{u} \leq \frac{1}{M_0^{k-1}} \inf_{\tilde{Q}_{2^{-i}}} u = 1$;
- $\mathcal{L}\tilde{u} \leq \frac{f(x.y)}{M_0^{k-1}} = \hat{f}(y)$, $\|\hat{f}\|_{L^n} \leq \delta_0$.

$$\xrightarrow[\tilde{u}+\varphi]{ABP} |\{\tilde{u} \leq M_0\} \cap Q_1| \leq \mu_0 |Q_1|$$

$$\Rightarrow |\{\tilde{u} \leq M_0^k\} \cap Q_{2^{-i}}| \geq \mu\, Q_{2^{-i}}$$

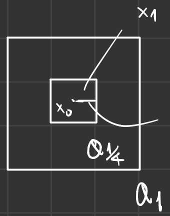
$$\Downarrow$$

$$|\{\tilde{u} > M_0^k\} \cap Q_{2^{-i}}| \leq (1-\mu_0)Q_{2^{-i}}$$

矛盾！

Recall. $u \geq 0$. in $Q_{4\sqrt{n}}$. $\inf_{Q_{1/4}} u \leq 1$. $\mathcal{L}u = f$, $\exists\, \delta_0.\ k_0$ s.t. $\|f\|_{L^n} \leq \delta_0 \Rightarrow \sup_{Q_{1/4}} u \leq k_0$

Proof :



$u(x_1) = (1+\gamma_0)k_0$

$l \cong (k_0)^{-\varepsilon_0/n}$

$\exists\, x_2 \in Q_{l_1}(x_1)$, $l_1 \cong \left[(1+\gamma_0)k_0\right]^{-\frac{\varepsilon_0}{n}}$

$\vdots$

$x_{j+1} \in Q_{l_j}(x_j)$. $l_j \cong \left[(1+\gamma_0)^j k_0\right]^{-\frac{\varepsilon_0}{n}}$

$$|x_N - x_0| \leq \sum |x_{j+1} - x_j| \cong k_0^{-\frac{\varepsilon_0}{n}} < \frac{1}{4}$$

如果 $u(x_N) = (1+\gamma_0)^N k_0$ $N \to +\infty$. 爆掉了！ 矛盾.

$\therefore$ 一开始找不到 $x \in Q_{1/4}$. ⋯

$\exists$ universal k_0 . $\gamma_0 = \gamma_0(\varepsilon_0)$ (independent of k_0). s.t.

if $\boxed{u(x_0) = k_0}$ for some $x_0 \in Q_{1/4}$, then $\sup_{Q_l(x_0)} u \geq (1+\gamma_0)k_0$

$l \cong (k_0)^{-\varepsilon_0/n}$

Look at two sets in $Q_{\frac{l}{4\sqrt{n}}}(x_0)$

$A = \{u \geq \frac{k_0}{2}\} \cap Q_{\frac{l}{4\sqrt{n}}}$ $B = \{u < \frac{k_0}{2}\} \cap Q_{\frac{l}{4\sqrt{n}}}$

By $L_W^{\varepsilon_0}$ - estimates. $|A| \leq C k_0^{-\varepsilon_0} < \frac{1}{2} |Q_{\frac{l}{4\sqrt{n}}}|$

Take $l \simeq (k_0)^{-\frac{\varepsilon_0}{n}}$

Consider $x = x_0 + \frac{l}{4\sqrt{n}} y$ $y \in Q_{4\sqrt{n}}$

$$V(y) = \frac{(1+\gamma)k_0 - u(x(y))}{\gamma k_0} \quad y \in Q_{4\sqrt{n}}$$

$y = 0$ 时. $V(0) = 1$

$\therefore \inf V(y) \leq 1$

Assume by contradiction argument. $\sup_{Q_\ell(x_0)} u \leq (1+\gamma_0)k_0 \Rightarrow V \geq 0$ in $Q_{4\sqrt{n}}$

$\mathcal{L} v = f$

By L^ε_W, $\left| \left\{ v(y) > \frac{\frac{1}{2}+\gamma}{\gamma} \right\} \cap Q_1 \right| \leq C_n \left(1 + \frac{1}{2\gamma}\right)^{-\varepsilon_0}$

$\rightsquigarrow \left| \left\{ u > \frac{k_0}{2} \right\} \cap Q_{\frac{\ell}{4\sqrt{n}}} \right| \leq C_0 \left(1+\frac{1}{2\gamma}\right)^{-\varepsilon} |Q_{\frac{\ell}{4\sqrt{n}}}| < \frac{1}{4} |Q_{\frac{\ell}{4\sqrt{n}}}|$

$|A| + |B| < |Q_{\ell/4\sqrt{n}}|$ 矛盾!

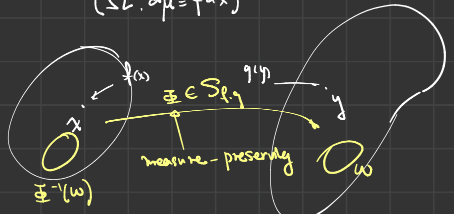
\#

Let $\Omega, \Omega^* \subseteq \mathbb{R}^n$ bounded, $f \in L^1(\Omega)$, $g \in L^1(\Omega^*)$, $f \geq 0$, $g \geq 0$ $(\Omega^*, d\mu^* = g\,dy)$

$(\Omega, d\mu = f\,dx)$

$$\int_\Omega f = \int_{\Omega^*} g$$

(mass. balance)



Let $C: \Omega \times \Omega^* \longrightarrow \mathbb{R} \longleftarrow$ cost

$\uparrow$ cost function

Consider $T \in S_{f,g} = \{ \Phi : \Omega \to \Omega^* \mid \Phi_\# f = g \}$

We say $\Phi_\# f = g$, if $\forall \omega \subseteq \Omega^*$ (Borel set)

$$\int_{\Phi^{-1}(\omega)} f = \int_\omega g$$

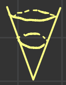
Monge's optional transport problem

Find a map $T_0 \in S_{p,q}$ s.t. $\mathcal{C}(T_0) = \inf_{\Phi \in S_{p,q}} \mathcal{C}(\Phi)$

where $\mathcal{C}(\Phi) = \int_\Omega c(x, \Phi(x)) f(x)\,dx \longleftarrow$ Total cost functional

History of existence.

$C_\varepsilon(x,y) = |x-y|^{1-\varepsilon}$



- 1781 G. Monge propose OT $c(x,y) = |x-y|$

1885, Appell direction of transport $x \rightsquigarrow Du(x)$ (Bordin Prize)

1940s. Kantnorwich linear programming <u>Nobel prize</u> (1975)

Sudakov 1979 $n=2$

Evans Gangbo 1999. $f, g \in C^{0,1}$, Ω, Ω^* are disjoint

Trunding - X.J. Wang 2001 / Caffarelli - Feldman - McCann 2002

Ambroso 2003 fix a gap of S. $n \geq 2$

- $c(x,y) = \frac{1}{2}|x-y|^2$ $(\longleftrightarrow c(x,y) - x \cdot y)$

Brenier 1991 ✓ optional $T(x) = Du(x)$, u is a convex function

$\Phi \in \mathcal{S}_{f,g}$ $\quad \int_{\Phi^{-1}(\omega)} f = \int_\omega g \quad \omega \subseteq \Omega^* \quad \Longleftrightarrow \quad \int_\Omega f h \circ \Phi = \int h g$

$\uparrow \int_\Omega f 1_\omega \circ \Phi$ $\qquad \int_{\Omega^*} 1_\omega g$

$\forall h \in C(\Omega^*)$

$(\because 1_\omega$ 可用 continous function逼近 !)

$$\mathcal{C}(\Phi) = \int_\Omega \frac{1}{2}|x - \Phi(x)|^2 f = \frac{1}{2}\int_\Omega |x|^2 f(x) - \int_\Omega x \cdot \Phi(x) f + \underline{\frac{1}{2}\int_\Omega |\Phi(x)|^2 f}$$

$$\| \quad \frac{1}{2}\int_{\Omega^*} |y|^2 g$$

Take $h = |y|^2/2$

<u>Thm</u> Suppose $\Omega, \Omega^* \subseteq \mathbb{R}^n$ bounded, $f, g \geq 0$ $f, g \in L^1$, $\int_\Omega f = \int_{\Omega^*} g$

Suppose $c : \mathcal{U} \to \mathbb{R}$, $\mathcal{U} \supseteq \bar{\Omega} \times \bar{\Omega}^*$ open, $c \in C^1(\mathcal{U})$ and

$\forall x \in \Omega$. $D_x c(x, \cdot) : \Omega^* \longrightarrow D_x c(x, \Omega^*) \subseteq \mathbb{R}^n$

homeomorphism

Then $\exists T_0 \in S_{f,g}$ s.t $\mathcal{C}(T_0) = \inf_{\Phi \in S_{f,g}} \mathcal{C}(\Phi)$

Moreover, $T_0(x) = [D_x c(x, \cdot)]^{-1}(-Du)$. u is c-convex function.

Kantinorich's dual functional

$$J(\varphi,\psi) = \int_{\Omega} (-\varphi) f + \int_{\Omega^*} (-\psi) g \,, \qquad \mathcal{K} = \{ (\varphi,\psi) \in C(\bar{\Omega}) \times C(\bar{\Omega}^*)$$

$$-\varphi(x) - \psi(y) \le c(x,y) \}$$

Kantorovich's dual problem

Find $(u,v) \in \mathcal{K}$ s.t. $J(u,v) = \sup_{\mathcal{K}} J(\varphi,\psi)$

Def We say $u \in C(\Omega)$ is c-convex (indexed by Ω^*) if $\exists\, v \in C(\Omega^*)$ st.

$$u(x) = \sup_{y \in \Omega^*} \{ \underline{-c(x,y) - v(y)} \} = v^c \leftarrow \text{c-transform}$$

若取 $x \cdot y - v(y) = l_y(x)$

Thm1 · Let Ω, Ω^* f, g c be as in Thm.

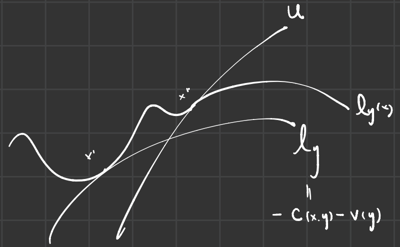
· Let $u = v^c$ be a c-convex function, $T_u(x) = [D_x c(x,\cdot)]^{-1} (D u(x))$

The following are equivalent (TFAE)

(i) (u,v) are potentials of J $\left(J(u,v) = \sup_{\mathcal{K}} J(\varphi,\psi) \right)$

(ii) $T_u \in S_{f,g}$

(iii) T_u is opeional $\left(\mathcal{C}(T_u) = \inf_{\Phi \in S_{f,g}} C(\Phi) \right)$



Define $\chi_u(x) = \{ y \in \bar{\Omega}^*,\ u(x) + v(y) = -c(x,y) \}$

Remark

(i) $\forall x \in \Omega^*$, take $y \in \chi_u(x)$, then $u(\cdot) + c(\cdot, y)$ achieve global minmum

(ii) If u is differentiable at x, then by (i)

$$0 = Du(x) + D_x c(x,y) \Leftrightarrow y = [D_x c(x,\cdot)]^{-1} (-Du(x))$$

(iii) $\forall x', x \in \Omega$, $y' \in \chi_u(x')$, $y \in \chi_u(x)$

By (i) $u(x') + c(x',y) \geq u(x) + c(x,y)$

$u(x) + c(x,y') \geq u(x') + c(x',y')$

$$\frac{c(x',y') - c(x,y')}{|x-x'|} \le \frac{u(x) - u(x')}{|x-x'|} \le \frac{c(x',y) - c(x,y)}{|x-x'|}$$

$\mathcal{U} \supseteq \bar{\Omega} \times \bar{\Omega}^*$

$$\Longrightarrow \|u\|_{Lip(\Omega)} \le \sup_{y} \|D_x c(\cdot,y)\|_{Lip(\Omega)} \le C \qquad u \in C^1(\mathcal{U})$$

Radenachers

proof of thm1 For any $(\varphi.\psi) \in \mathcal{K}$, $T \in S_{f.g}$, $J(\varphi.\psi) = \int_{\Omega} -\varphi f + \int_{\Omega^*} -\psi g$ $\left(\int_{\Omega} f h\circ T = \int_{\Omega^*} g h \quad \forall h \in C(\Omega^*)\right)$

$$= \int_{\Omega} -\varphi f + \int_{\Omega} (-\psi\circ T) f$$

$$= \int_{\Omega} -[\varphi(x) + \psi(Tx)] f$$

$$\leq \int_{\Omega} c(x.Tx) f(x)\, dx$$

$$= \mathcal{C}(T).$$

$$\Longrightarrow \sup_{\mathcal{K}} J(\varphi.\psi) \leq \inf_{S_{f.g}} \mathcal{C}$$

(ii) $\Rightarrow$ (iii) $(\overset{}{u}.v) \in \mathcal{K}$, $T_u \in S_{f.g}$, $u \overset{\shortparallel}{} v^c$

$$J(u.v) = \int -(u(x) + v(\underset{\overset{\shortparallel}{y \in \chi_u(x)}}{T_u x})) f$$

$$= \int c(x.T_u(x)) f$$

$$= \mathcal{C}(T_u)$$

$$J(u.v) \leq \sup_{K} J \leq \inf_{S_{f.g}} \mathcal{C} \leq \mathcal{C}(T_u) = J(u.v) \Big| \leq \sup_{K} J$$

全取等

(iii) $\Rightarrow$ (i) ✓

(i) $\Rightarrow$ (ii) Fix $h \in C(\Omega^*)$ and $\varepsilon \in (0.1)$

Consider $v_\varepsilon(y) = v(y) + \varepsilon h(y)$

$$u_\varepsilon(y) = \sup_{y \in \Omega^*} \{-c(x.y) - v_\varepsilon(y)\} = v_\varepsilon^c$$

We want to $u_\varepsilon(x) = u(x_0) - \varepsilon h(T_u(x_0)) + o(\varepsilon)$ if u is differentiable at x_0.

On one side, take $y_0 = \chi_u(x_0) = T_u(x_0)$

$$0 = u(x_0) + v(y_0) + c(x_0.y_0)$$

$$= u(x_0) + \underset{\sim\sim}{V_\varepsilon(y_0)} - \varepsilon h(y_0) + \underset{\sim\sim}{c(x_0.y_0)}$$

$$\geq \underline{u(x_0) - u_\varepsilon(x_0)} - \varepsilon h(y_0)$$

On the other hand take $y_\varepsilon \in \chi_{u_\varepsilon}(x_0) \rightsquigarrow u_\varepsilon(x_0) + v_\varepsilon(y_\varepsilon) = -c(x.y_\varepsilon)$

$\downarrow$ x_0处可微

$\underset{\sim\sim}{y_\varepsilon \to y_0}$

$$0 \leq u(x_0) + v(y_\varepsilon) + c(x_0.y_\varepsilon)$$

$$= u(x_0) + v_\varepsilon(y_\varepsilon) - \varepsilon h(y_\varepsilon) + c(x_0.y_\varepsilon)$$

$$= \underline{u(x_0) + [-u_\varepsilon(x_0)]} - \varepsilon h(y_\varepsilon)$$

$$\underset{v_\varepsilon^{c}}{} \urcorner^{x} \quad \underset{v+\varepsilon h}{} \quad = u(x_0) - u_\varepsilon(x_0) - \varepsilon h(y_0) + \varepsilon\underbrace{(h(y_0) - h(y_\varepsilon))}_{o(1)}$$

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} J(\overset{\|}{u_\varepsilon}, \overset{\|}{v_\varepsilon})$$

$$= \frac{d}{d\varepsilon} \int -u_\varepsilon f + \int -v_\varepsilon g$$

$$= \int_\Omega h(T_u) f - \int_{\Omega^*} hg \qquad \square$$

proof existence of OT. $\longleftarrow T_u(x) = |D_x c(x,\cdot)|^{-1}(-Du(x)) \;/\; \chi_u = \{y \mid u(x)+v(y) = c(x,y)\}$

It suffices to show $\exists (u, v)$, $u(x) = v^c(x) := \sup_{y\in\Omega^*} \{-c(x,y) - v(y)\} \leftarrow$ c-transform

s.t. $J(u,v) = \sup_{(\varphi,\psi)\in K} J(\varphi,\psi)$

where $J(\varphi,\psi) = \int_\Omega (-\varphi) f + \int_{\Omega^*} (-\psi) g \quad K = \{(\varphi,\psi) \in C(\bar\Omega)\times C(\bar\Omega^*) : -\varphi(x) - \psi(x) \le c(x,y)$
$\forall x\in\Omega, y\in\Omega^*\}$

Let (u_k, v_k) be a maximizing sequence

$$J(u_k, v_k) \longrightarrow \sup_{(\varphi,\psi)\in K} J(\varphi,\psi)$$

Let $(\varphi,\psi)\in K$. $-c(x,y) - \psi(y) \le \varphi(x) \rightsquigarrow \sup_{y\in\Omega^*}\{-c(x,y) - \psi(y)\}$
$= \psi^c(y) \le \varphi(x)$

$\rightsquigarrow$

(ψ^c, ψ). $J(\psi^c, \psi) = \int -f\psi^c + \int -g\psi$
$\ge \int -f\varphi + \int -g\psi = J(\varphi,\psi)$

$\rightsquigarrow$

$(\psi^c, \underbrace{\psi^{cc}(y)})$ $\quad J(\psi^c, \psi^{cc}) \ge J(\psi^c, \psi) \ge J(\varphi,\psi)$

$= \sup_{x\in\Omega}\{-c(x,y) - \psi^c(x)\}$

We find ~~that~~ $(v_k, v_k) \rightsquigarrow (v_k^c, v_k^{cc})$ still maximizing sequence

c-convex

Consider $u_k^* = v_k^c - v_k^c(x_0)$ fix. an $x_0\in\Omega$

$v_k^* = v_k^{cc} + v_k^c(x_0) \longleftarrow (v_k^* = (u_k^*)^c)$

(u_k^*, v_k^*) is maximizing sequence.

$$J(u_k^*, v_k^*) = \int_\Omega (-v_k^c + v_k^c(x_0)) f + \int_{\Omega^*} (-v_k^{cc} - v_k^c(x_0)) g$$

$$\int_\Omega f = \int_{\Omega^*} g \curvearrowright = \int_\Omega -v_k^c f + \int_{\Omega^*} -v_k^{cc} g = J(v_k^c, v_k^{cc})$$

We have $\|u_k^*\|_{Lip(\Omega)} \le C$, $\underline{\|u_k^*\|_{L^\infty(\Omega)} \le C}$

$\le \sup_{y\in\Omega^*} \|D_x c(\cdot, y)\|_{L^\infty(\Omega)}$,

$\Leftarrow$ $u_k^*(x) \le \overset{0}{\cancel{u_k^*(x_0)}} + \|u_k^*\|_{Lip} \underbrace{\text{diam}\,\Omega}_{\text{bounded.}}$

$\|v_k^*\|_{Lip(\Omega^*)} \le C$, $\|v_k^*\|_{L^\infty(\Omega)} \le C$

$v_k^* = (u_k^*)^c$

__AA Thm__

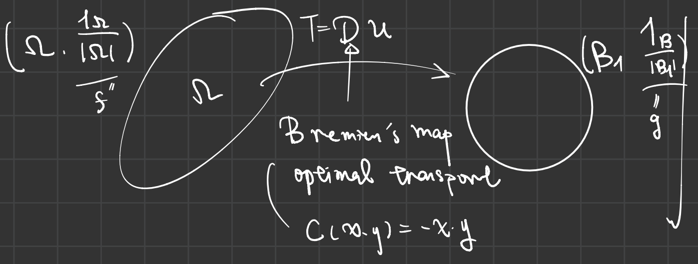
$\rightsquigarrow$ $(u_k^*, v_k^*) \to (u, v)$. $J(\underbrace{u, v}_{\in \mathcal{K}}) = \sup_{\mathcal{K}} J(u_k^*, v_k^*)$ $\square$

$(J(\underset{\underset{u}{\|}}{v^c}, v) \ge)$

Rmk: (u, v) maximizer, $(\tilde u, \tilde v)$ maximizer | $\exists c$. $\tilde u = u + c$, $\tilde v = v + c$

const

Isoperimetric inequality (Argument due to Gromov Trudinger)



$c(x, y)$

$$Tu = [D_x c(x, \cdot)]^{-1}(-Du(x))$$

c-convex

$D_x c(x,y) = -y$.

$Tu = Du(x)$

(a) $|Du(x)| \le 1$

(b) $\det D^2 u = \frac{|B_1|}{|\Omega|}$

$T \in S_{f,g}$, $\forall\, \omega^* \subseteq \Omega^*$ Borel set $\int_{\underbrace{T^{-1}(\omega^*)}_{=:\,\omega}} f = \int_{\omega^*} g.$

$$\int_\omega f = \int_{\omega^*} g = \int_{Du(\omega)} g \underset{\substack{\uparrow \\ \text{area formula}}}{=} \int_\omega \det D^2u \; g(\nabla u) \rightsquigarrow f = \det D^2u \; g(\nabla u)$$

$$\frac{|B_1|^{\frac1n}}{|\Omega|^{\frac1n}}|\Omega| = \int_\Omega (\det D^2u)^{\frac1n} \underset{(\lambda_1\cdots\lambda_n)^{1/n} \le \frac1n(\lambda_1+\dots+\lambda_n)}{\le} \int_\Omega \frac1n \Delta u = \frac1n \int_{\partial\Omega} \langle Du, \nu\rangle \le \frac1n |\partial\Omega|$$

✓

Rigidity . If Ω attains isoperimetric inequality $\rightsquigarrow \Omega = B_r(x_0)$

(i) Ω is connected $\left(\Omega = \Omega_1 \cup \Omega_2 ,\ |B_1|^{\frac1n}\left(|\Omega_1|+|\Omega_2|\right)^{1-\frac1n} = \frac1n\left(|\partial\Omega_1| + |\partial\Omega_2|\right)\right)$

$\underbrace{(a+b)^{1-\frac1n} \geq a^{1-\frac1n} + b^{1-\frac1n}}$ "a=0 or b=0" $\geq |B_1|^{\frac1n}\left(|\Omega_1|^{1-\frac1n} + |\Omega_2|^{1-\frac1n}\right)$

$D^2u \sim (\lambda_1 \cdots \lambda_n)$ $\quad \lambda_1 = \dots = \lambda_n$

$$D^2u = \frac{\Delta u}{n} I = \frac{|B_1|^{1/n}}{|\Omega|^{1/n}} I$$

$$\rightsquigarrow D\left(Du - \frac{|B_1|^{1/n}}{|\Omega|^{1/n}}x\right) = 0 \implies T = Du(x) = \overbrace{\frac{|B_1|^{1/n}}{|\Omega|^{1/n}}}^{=\,r} x - x_0$$

Ω is connected

" $\Omega = T^{-1}(B_1) = B_r(x_0)$

$$|\Omega + \Omega^*|^{\frac1n} \geq |\Omega|^{\frac1n} + |\Omega^*|^{\frac1n} \qquad \Omega + \Omega^* = \{x+y : x\in\Omega,\ y \in \Omega^*\}$$

Remark: Take $\Omega^* = \varepsilon B_1$, $|\Omega + \varepsilon B_1|^{1/n} \geq |\Omega|^{1/n} + \varepsilon |B_1|^{1/n}$

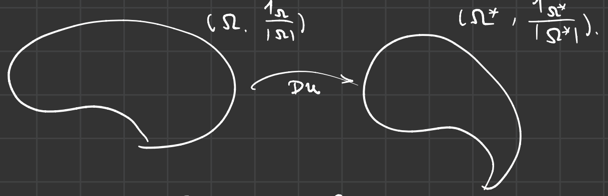
$\qquad\qquad\qquad\qquad\qquad\quad$ ‖

$$|\Omega_\varepsilon|^{\frac1n} = |\Omega|^{\frac1n} + \square\,\varepsilon + o(\varepsilon)$$

$$\frac1n|\Omega|^{\frac1n - 1}\underbrace{\frac{d}{d\varepsilon}|\Omega_\varepsilon|}_{|\partial\Omega|}$$

$$\implies \frac1n|\Omega|^{\frac1n-1}|\partial\Omega| \geq |B_1|^{1/n}$$

✓



$$|\Omega + \Omega^*| \geq \int_{(Id+Du)(x)} dy \geq \int_\Omega \det(I + D^2u)\, dx \geq \iint_\Omega \left[1 + \det^{\frac{1}{n}} D^2u\right]^n = \left(1 + \frac{|\Omega^*|^{1/n}}{|\Omega|^{1/n}}\right)|\Omega|^{1/n}$$

$\forall y$

$y = Du(x)$ 有限制 又y.

$\det(A+B) \geq \det^{\frac{1}{n}} A + \det^{\frac{1}{n}} B$

$(A, B \geq 0)$ (Exercise)

$\square$

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3. Geometric optics

$$c(x,y) = \begin{cases} -\log x\cdot y & x, y \in S^n \leftarrow \text{Aleksandrov (1940s)} \\ \log(1 - x\cdot y) & x, y \in S^n \leftarrow \text{Reflector design (X.-J. Wang)} \\ \log(1 - \kappa x\cdot y) & \kappa < 1 \\ -\log(\kappa x\cdot y - 1) & \kappa > 1 \end{cases}$$

Refraction design (Gutiérrez – Huang 2009)

Villani, Old and New. OT.

Glaudo - Figalli An invitation to OT Wasserstein distance and gradient flows.

$$Tu(x) = [D_x c(x, \cdot)]^{-1}(-Du(x)) \longleftrightarrow D_x c(x, Tu(x)) = -Du(x)$$

$$c_i(x, Tu(x)) = -u_i$$

$$c_{ij}(x, Tu(x)) + c_{i,y_j} T^j_i = -u_{ij}$$

$$\Rightarrow D^2u + D_x^2 c = -D^2_{xy} c\, DT$$

$$\Rightarrow \det(D^2u + D_x^2 c(x, Tu(x))) = |\det D^2_{xy} c(x, Tu(x))| \frac{f(x)}{g(Tu(x))}$$

area formula

Jac T(x)

$T\# f = g$

Monge - Ampère

$D^2u(x) + D_x^2 c(x,y) \geq 0$ attains minimal at x

$y \in \chi_u(x)$ $\quad y = Tu(x)$

- $c(x,y) = -x\cdot y$ $\qquad \det D^2u = \frac{f(x)}{g(Du(x))}$, $\quad Du(x) = \Omega^*$

(i) Delanoe $(n=2)$ Urbas $(n\geq 2)$. uniformly convex domains : $f,g>0 \in C^{1,1}$
$u \in C^{3,\alpha}(\bar\Omega)$

(ii) Caffarelli uniformly convex $f,g \in C^\alpha(\bar\Omega) \Rightarrow u \in C^{2,\alpha}(\Omega)$

(iii) C. Chen. J.K. Liu – X.J. Wang convex domains $f,g \in C^\alpha(\bar\Omega) \Rightarrow u \in C^{2,\alpha}(\Omega)$

$n=2$.

$$\det D^2u \overset{\partial}{=} \begin{vmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{vmatrix} = u_{11}u_{22} - u_{12}u_{21} = \underset{\frac12 u_{22}}{\underset{\downarrow}{a_{11}}} u_{11} + \underset{\frac12 u_{11}}{\underset{\downarrow}{a_{22}}} u_{22} + \underset{-\frac12 u_{12}}{\underset{\downarrow}{a_{12}}} u_{12} + \underset{-\frac12 u_{21}}{\underset{\downarrow}{a_{21}}} u_{21} = \overset{C^\alpha}{\underbrace{\sum_{1\leq i,j\leq 2} a_{ij}u_{ij}}_{\text{uniformly elliptic}}}$$

$$a_{ij} = \begin{pmatrix} \frac12 u_{22} & -\frac12 u_{12} \\ -\frac12 u_{21} & \frac12 u_{11} \end{pmatrix} \qquad |D^2u| \leq C$$

- General $c(x,y)$ $\qquad \det(D^2u - A(x,Du)) = \varphi(x, Du(x))$
$\qquad A(x,Du) = -C_{xx}(x, Tu(x))$

X.N. Ma – Trudinger – X.J. Wang $\quad D^2_{p_ip_j} A_{kl}\, \xi_i \xi_j \eta_k \eta_l \geq C_0 |\xi|^2 |\eta|^2 \quad \xi \perp \eta$
$(C_0 > 0)$
$C = \frac12 d^2(x,y)$

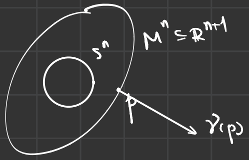
Loeper necessary

Figalli, Kim, McCann. J.K. Liu., Trudinger, X.-J. Wang ...

$C = |x-y|$ $\quad \Omega, \Omega^* \subseteq \mathbb{R}^2$. T is continuous, $T \notin C^{\frac23+\varepsilon}$ $\forall \varepsilon > 0$.

(Colombo Indrei $T \notin C^{\frac12+\varepsilon}$) $\quad T \in \mathrm{Lip}(\Omega \setminus \text{null set})$. $\boxed{n=2}$

Minkowski problem (presenting Gauss curvature of convex hypersurface)
“预定”



$$\nu : M^n \longrightarrow S^n$$
$$p \longmapsto \nu(p)$$

Given $f \in C^\infty(S^n)$ Gauss curvature.

$\nexists\, M \subseteq \mathbb{R}^{n+1}$ s.t $K(\nu^{-1}(x)) = f(x)$

- Minkowski (1900s) discrete measure version
- Fenchel – Jesen Aleksandrov variational problem
- Mirenberg Pogrelow (1950s) $n=2$
- Cheng – Yau (1970s) Pogorelov $n \geq 2$

Minkowski problem Nirenberg Pogrelew . Cheng – Yau

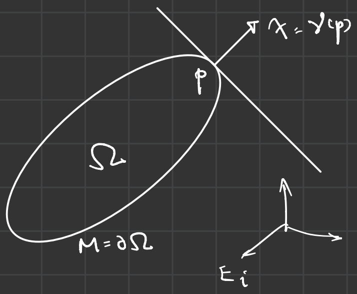
<u>Thm</u> If $f \in C^2(S^n)$, $f > 0$ and $\int_{S^n} \frac{x^i}{f} dx = 0$ $x=(x^1, x^2, \dots, x^{n+1})$

then $\exists!$ uniformly convex closed hypersurface $M^n \subseteq \mathbb{R}^{n+1}$ $C^{3,\alpha}$, $\alpha \in (0,1)$

s.t $K(\nu^{-1}(x)) = f(x)$ $\forall x \in S^n$

Remark: $K = \prod_{i=1}^{n} \kappa_i$ κ is the eigenvalue of h_{ij} w.r.t. g_{ij}, $\det(h_{ij} - \kappa g_{ij}) = 0$.

$$= \frac{\det h_{ij}}{\det g_{ij}}$$



This condition is necessary

Gauss map's Jacobi

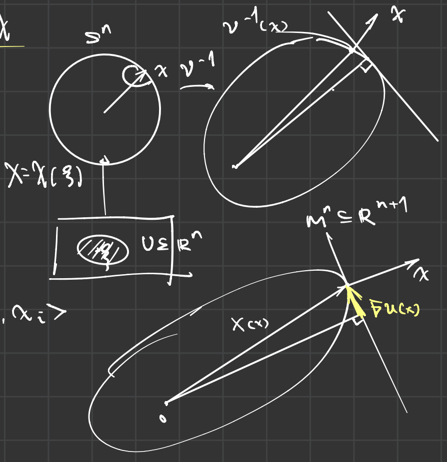
$K = \mathrm{Jac}\,\nu$

$$\int_{S^n} \frac{x^i}{f} dx = \int_{S^n} \frac{x^i}{K} \boxed{\frac{dx}{dp}} dp = \int_M \nu \cdot E_i \, d\mu$$
$$= \int_\Omega \operatorname{div} E_i$$
$$= 0$$

Define $u : S^n \longrightarrow \mathbb{R}$, $x \longmapsto u(x) = \underset{X(x)}{\underline{\langle \nu^{-1}(x), x \rangle}} = \sup_{z \in \Omega} z \cdot x$

带 (¯) 指球面上的量

Lemma: If $X : S^n \longrightarrow M^{\subseteq \mathbb{R}^{n+1}}$, $x \longmapsto \nu^{-1}(x)$ $\quad X(x) = \underbrace{\bar\nabla u(x)}_{\text{切向}} + \underbrace{u(x)\cdot x}_{\text{法向}}$



- $u_i = \cancelto{0}{\langle X_i, x\rangle} + \langle X, x_i\rangle$

- Assume $X = A^k \underset{\substack{\cap \\ x^\perp}}{x_k} + u(x)\, x$

$$\langle X, x_i\rangle = A^k \underbrace{x_k \cdot x_i}_{\bar g_{ik}} \Longrightarrow A^k = \bar g^{ki} \langle X, x_i\rangle$$
$$= \bar g^{ki} u_i$$

$$X = \underbrace{\bar g^{kl} u_i X_k}_{\parallel\ \bar\nabla u} + u(x) X$$

□

Lemma $\dfrac{1}{K(\nu^{-1}(x))} = \dfrac{\det(\bar\nabla^2 u + u\bar g)}{\det \bar g}(x) \quad \left(\Longrightarrow \dfrac{\det(\bar\nabla^2 u + u\tilde g)}{\det \tilde g} = \dfrac{1}{f} \right)$

$x_i = x_{\xi_i}$ (并非单位向量!)

Proof $u_{ij} = \langle X, \underbrace{x_{ij}}_{\substack{\parallel \\ \bar\Gamma_{ij}^k x_k - \bar h_{ij} x}} \rangle + \langle X_j, \underbrace{x_i}_{h_i^k X_k} \rangle$

S^n Gauss formula

M^n: Weingarten formula

$$= \Gamma_{ij}^k u_k - \bar g_{ij} u + \underbrace{h_i^k g_{kj}}_{= h_{ij}}$$

umbilic

全脐

$$\Longrightarrow \boxed{\bar\nabla^2_{ij} u + u \bar g_{ij} = h_{ij}}$$

$$\bar g_{ij} = \langle x_i, x_j\rangle = \langle h_i^k X_k, h_j^l X_l\rangle = h_i^k g_{kl} h_j^l = h_{ik} h_j^k = h_{ik} h_{jl} \underset{\substack{\downarrow \\ \text{inverse of } (g_{lk})}}{\bar g^{lk}}$$

$$\rightsquigarrow \frac{\det(\bar\nabla u + u\bar g)}{\det(\bar g)} = \frac{\det h}{(\det h)^2 (\det g)^{-1}} = \frac{\det g(x)}{\det h} = \frac{1}{K(\nu^{-1}(x))}$$

□

Corollary. Main theorem $\Longleftrightarrow$ Find $u \in C^{3,\alpha}(S^n)$, $\bar\nabla^2 u + u\bar g > 0$

s.t. $\dfrac{\det(\bar\nabla^2 u + u\bar g)}{\det \bar g} = \dfrac{1}{f}$ $\quad (*)$

We solve (*) by continuity method

$x = (x^1, \cdots, x^{n+1}) \in S^n$

$\forall i$

Define $\psi : S_m \longrightarrow S_{m-\alpha}$

$S_m = \{ w \in H^m(S^n), \int_{S^n} w\, x^i dx = 0 \}$

$w \longmapsto \psi(w) = \det(\bar\nabla^2 w + w I)$

$\int x^k \det(\underbrace{\bar\nabla^2 w + w I}_{W}) \neq 0$

$\frac{1}{n}\psi^{ij}(W_{ij} + w\delta_{ij})$

integral by parts

$$\frac{1}{n}\int \big(\underbrace{x^k_{;ij}}_{-\bar g_{ij}x^k} + x^k\delta_{ij}\big)\psi^{ij} w$$

∴ it's well-defined.

$\lambda_1 \cdots \lambda_n$

$\Sigma\, \lambda_1 \cdots \lambda_{n-1} \cdot \lambda_n$

$\psi^i : \lambda_1 \cdots \lambda_n \downarrow \lambda_1 \cdots \lambda_{n-1}$

$\psi^i W_i : \lambda_1 \cdots \lambda_{n-1} \downarrow \lambda_1 \cdots \lambda_{n-1} \lambda_n$

求3-9 协变导数

$W_{ijk} = W_{ikj} = W_{kji}$

Define $I = \{ t \in [0,1],\ \psi(u) = 1 - t + t\frac{1}{f}$ admits $C^{3,\alpha}$ uniformly convex solution $\}$

(i) $0 \in I$

(ii) I is open. $\exists t_0 \in I$, $u = u^{t_0}$, $\psi(u) = 1 - t_0 + t_0/f = f_{t_0}$ $\exists t$ is closed to t_0

(iii) I is closed.

$$\mathcal{L}_{u^{t_0}}(\phi) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\psi(u^{t_0} + \varepsilon\phi) = \psi^{ij}_{u^{t_0}}(\bar\nabla^2_{ij}\phi + \phi\bar g_{ij})$$

线性化算子.

↓

可验证自伴性.

隐函数定理. $S_m \longrightarrow S_{m-j}$

$\mathrm{Ker}\,\mathcal{L}_w \overset{\text{span}}{=} \{x^1 \cdots x^{n+1}\}$

$\mathrm{Range}\,\mathcal{L}_w = \{\mathrm{span}(x^1 \cdots x^{n+1})\}^\perp$

$\exists t_j$ s.t. $\det(\bar\nabla^2 u^{t_j} + u^{t_j} I) = f_{t_j}$

$t_j \to t_*$ $\Leftarrow$ $\boxed{\|u^{t_j}\|_{C^{2,\alpha}(S^n)} \le C}$ (integent of j)

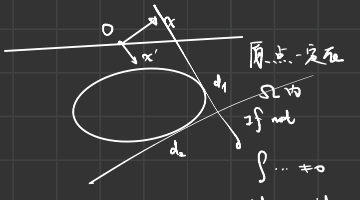
$u^{t_j} \longrightarrow u_\alpha$

C^0-estimate

$M \longrightarrow M + \vec a$, $u \rightsquigarrow u_a = u(x) + a X$

$\exists\, \underset{(a_1, \cdots, a_{n+1})}{\vec a} \in \mathbb{R}^{n+1}$ s.t. $\int_{S^n} u_a(x)\, x^i = 0$ $\left(0 = \int_{S^n} u x^i dx + \int_{S^n} \underset{a_k x^k x^i}{(a \cdot x) x^i} \right)$

We always assume $\int_{S^n} u(x) x^i dx = 0$ $\forall i$



原点一定在 Ω内

If not

$\int \cdots \neq 0$

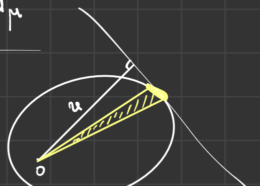
$\because |d_1 O| < |d_2 O|$

$$|\Omega| = \frac{1}{n+1}\int_{S^n}\frac{u}{K} = \int_{S^n}\frac{u}{f}\frac{1}{n+1}, \quad \frac{1}{K}dx = d\mu$$

$$\geq C\int_{S^n} u$$

$$= C\int_{\{x\cdot x_0 < 0\}\cup\{x\cdot x_0 \geq 0\}} u$$

Assume $u(x_0) = \max_{S^n} u$



$$\boxed{\begin{aligned}&\int_{S^n}\frac{1}{K}\,dx=\int_{S^n}\frac{1}{f}\\&\quad\ \|\\&\int_M d\mu=|M|\end{aligned}}$$

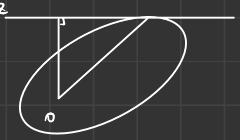
$$\geq C\int_{\{x\cdot x_0\geq 0\}}u$$

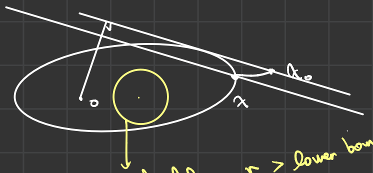
$$\geq C\,u(x_0)\int_{S^n}(x\cdot x_0)_+$$

$$\geq C'\,u(x_0)$$

$u(x)\geq u(x_0)\,x_0\cdot x$

$u(x)=\sup_z x\cdot z$





∃ ball r > lower bonded

撑起来

$$\frac{1}{2}d\leq u(x_0)\leq C|\Omega|\leq C\,|\overset{M}{\overset{\|}{\partial\Omega}}|^{\frac{n}{n-1}}\leq C_n f$$

(等周不等式)

$$0\leq \min\leq\max\leq C$$

$$d\leq C|\Omega|\leq C f_-\,d^n\Rightarrow f_-\geq\frac{C}{d^{n-1}}\geq C'$$

δ (strictly lower bonded)

C^1 – estimates

$$X=\bar\nabla u+uX\Rightarrow|\bar\nabla u|\leq\frac{|x|}{d}+\frac{|uX|}{\|u\|_{L^\infty}}\leq C$$

C^2 – estimates (using maximum principle)

$$\|\bar\nabla^2 u\|\leq C\iff\max_{S^n}\lambda_{max}\leq C$$

λ_{max} is the target eigen values of $\bar\nabla^2 u+uI$ ← $(\lambda_1,\dots,\lambda_n)$

$$\downarrow$$

$$\bar\nabla^2 u+uI\geq\delta_0 I$$

$$\text{min}\rightarrow\underline{\lambda_1}\cdots\overset{\to 0}{\lambda_n}=1/f\cong 1$$

↓ 0 max

Consider
$$\boxed{\begin{aligned}G&=\sum\lambda_i\\&=\Delta_{S^n}u+nu\end{aligned}}$$

$u_{kk,i}=u_{ki,k}+\bar R^h_{kki}u_h$: Ricci identity

$$G_i=(u_{kk}+nu)_i=u_{kki}+nu_i$$

$$\begin{aligned}G_{ij}&=u_{kk,ij}+nu_{ij}\\&=u_{ki,kj}+\bar R^h_{kki}u_{hj}+nu_{ij}\\&=u_{kijk}+\cdots\\&=u_{ij,kk}+\cdots\end{aligned}$$

$$\log\det\underbrace{(u_{ij}+u\delta_{ij})}_{b_{ij}}=\hat f$$

$$\tilde b^{ij}\cdot(u_{ij,k}+u_k\delta_{ij})=\hat f_k$$

$$(\log\det b_{ij})'=\frac{\partial\log\det b_{ij}}{\partial b_{ij}}\cdot b_{ij,k}$$

$\tilde b^{ij}$

$$G_{ij} = \Delta_{S^n} b_{ij} + G\,\bar{g}_{ij} - n b_{ij}$$

$$0 \geqslant \tilde{b}^{ij} G_{ij} = \tilde{b}^{ij} \Delta_S b_{ij} + G\,\tilde{b}^{ij}\hat{g}_{ij} - n^2$$

$$\geqslant -C_0 + G(x)\underbrace{C_1}_{>0}$$

$$G(x) = \max G(x) \leqslant C_{n,f}$$

$$\tilde{b}^{ij} b_{jl} = \delta^i_l$$

$$\partial\tilde{b}\, b + \tilde{b}\,\partial b = 0$$

$$\therefore \partial\tilde{b} = -\tilde{b}\,\partial b\,\tilde{b}$$

再求导数.

$$\underline{\tilde{b}^{ij}\left(u_{ij,kk} + u_{kk}\delta_{ij}\right)}$$

$$- \tilde{b}^{ip}\tilde{b}^{jq}\frac{\partial b_{pq}}{\partial b_{rs}} b_{rs,k} \longrightarrow$$

$$\tilde{b}^{ir}\tilde{b}^{js} b_{rs,k} b_{ij,k} = \tilde{f}_{kk}$$

$$\Delta_{S^n} b_{ij}$$

$$\|u\|_{C^1(S^n)}^{\leq C} \xrightarrow{\quad} \|u\|_{C^{2,\alpha}(S^n)} \leq C$$

$$\uparrow \text{Evans - Krylov}$$

$$\log\det(\nabla^2 u + uI) = -\log f = \hat{f}$$

$$\tilde{b}^{ij}\left(\bar{\nabla}_{ij} u_e + u_e \delta_{ij}\right) = \hat{f}_e \Rightarrow u_e \in C^{2,\alpha}$$

"closed" □.